On spontaneous fission and -decay half-lives of atomic nuclei

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K Pomorski, M Warda and A Zdeb

Department of Theoretical Physics, Maria Curie-Skłodowska University, Lublin, Poland

E-mail: azdeb@kft.umcs.lublin.pl

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Abstract

It is shown, that the Gamow-like model with only one adjustable parameter–radius constant—is able to reproduce well the alpha-decay half-lives for all even–even nuclei with the proton number larger than 50. The systematics for odd–A and odd–odd isotopes can be also well described when one introduces an additional hindrance factor. A similar model based on the W J Świątecki idea from 1955 is developed to reproduce the spontaneous fission half-lives of transactinide nuclei. The achieved accuracy of reproduction of the data is better than that obtained in more advanced theories.

Keywords: nuclear fission, alpha-decay, fission barrier height, half-lives, gamow model, liquid drop model

(Some figures may appear in colour only in the online journal)

1. Introduction

Large progress in synthesis of heavy nuclei done in the last decades, mainly in laboratories in GSI-Darmstadt and JINR-Dubna (see e.g. reference list in the review paper [1] and in [2, 3]) raises new demands for a better and better theoretical description of their decay modes. Spontaneous fission and α-radioactivity are the most important processes of disintegration of heavy nuclei.

Alpha decay occurs most often in actinides region, but is also observed in isotopes with $Z \geqslant 52$. The first theoretical interpretation of the alpha-decay process was given independently by Gamow [4], Gurney and Condon [5] in 1928 year. Emission of the alpha particle is treated as quantum-mechanical tunnelling through the nuclear Coulomb barrier, where the probability of emission is calculated using one-dimensional WKB approximation. Recently it was proven that this approximation can be successfully used to evaluate the probability of tunnelling by the alpha particle and cluster as well (see [6]). It was also shown that within this simple model (containing only 1 adjustable parameter for even–even nuclei) one can reproduce alpha decay half-lives of heavy emitters with higher accuracy in comparison with the modern (containing 5 parameters) version [7] of Viola-Seaborg formula [8].

The observation in 1938 of the neutron-induced nuclear fission by Hahn and Strassmann came rather unexpected [9]. This new phenomenon was explained within a few weeks by Meitner and Frisch who established the most important features of low-energy fission: the energy released in this process equals almost 200 MeV and results from the Coulomb repulsion of the fission fragments and the number of neutrons emitted per fission event larger than one, that has opened a possibility for a chain reaction [10]. One and a half years after Hahn and Strassmann's discovery, Flerov and Petrzhak first observed the spontaneous fission of Uranium [11]. As we have written above, Gamow had explained the α-decay and its sometimes rather long half-lives by a quantum tunnelling process of a pre-formed α particle through the Coulomb barrier. So, according to the concepts of Meitner and Frisch, one could expect spontaneous fission of uranium also from the ground-state, but with a considerably longer half-life than for the α-decay, because of the larger reduced mass for almost symmetric fission.

The first quantitative estimates of the spontaneous fission probability became possible, when the microscopic-macroscopic model of the potential energy of deformed nuclei was developed and the Inglis cranking model was implemented to evaluate the inertia corresponding to the fission mode. Of course in time these models give more and more precise

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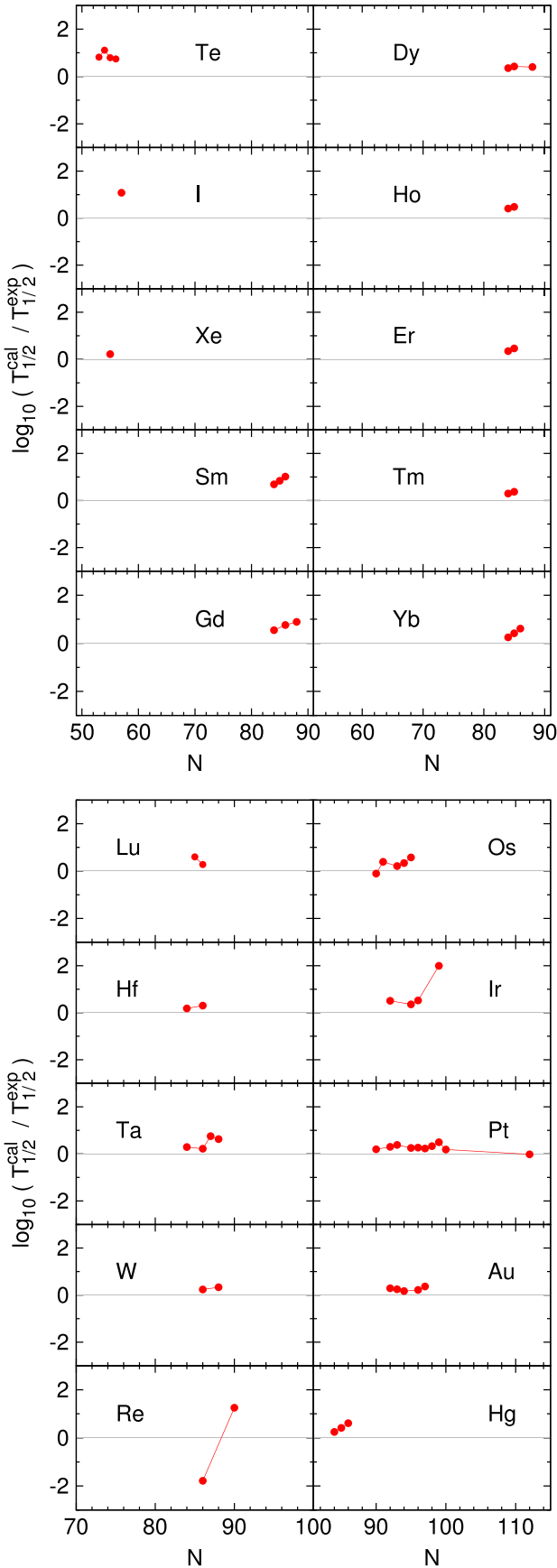
Figure 1. Differences of the theoretical and experimental [14] alpha decay half-lives on logarithmic scale, calculated for nuclei with $52 \leqslant Z \leqslant 70$ (top) and $71 \leqslant Z \leqslant 80$ (bottom).

reproduction of the spontaneous fission half-lives systematics and the fission mass distributions. A short critical description of the spontaneous fission theories will be presented in our paper in order to better understand the physical background of a simple model for the probability of this decay mode, which we have developed following an old idea of W. J. Świątecki. Namely, in 1955 he proposed a formula which was able to describe the global systematics of the spontaneous fission half-lives [12]. His simple phenomenological formula, based on correlations between logarithms of observed spontaneous fission half-lives and ground state microscopic corrections, reproduced well the experimental data known at that time. We are going to show in the following that his idea, combined with the modern version of the liquid drop model (LSD) [13], allows us to obtain satisfactory agreement with the spontaneous fission half-lives measured up to now.

The paper is organized as follows: the main assumptions of α-decay model and results obtained for emitters with $52 \leqslant Z \leqslant 110$ are presented in section 2. The WKB theory of the fission barrier penetration as well as the the semi-empirical Świątecki's formula for spontaneous fission half-lives will be described in section 3, where results for the isotopes with $90 \leqslant Z \leqslant 114$ are analyzed. Section 4 contains the summary.

2. Alpha decay

2.1. The model

The quantum tunnelling theory of alpha emission assumes that the decay constant λ is proportional to the barrier penetration probability P, frequency of assaults on the nuclear Coulomb barrier per time-unit n and particle preformation factor S_α. In the presented model the expression for decay constant is simplified as the effect of the preformation is effectively included into the probability P (see discussion in [6]):

$$\lambda = nP. \qquad (1)$$

The probability P of tunnelling through the barrier is calculated using one-dimensional WKB approximation:

$$P = \exp\left[-\frac{2}{\hbar} \int_R^b \sqrt{2\mu\left(V(r) - E_\alpha\right)} \, \mathrm{d}r \right], \qquad (2)$$

where μ is a reduced mass, R is the spherical square well radius

$$R = r_0\left(A_\alpha^{1/3} + A_d^{1/3}\right) \qquad (3)$$

and b is the exit point from the Coulomb barrier:

$$b = \frac{Z_\alpha Z_d e^2}{E_\alpha} . \qquad (4)$$

A_α, Z_α and A_d, Z_d are the mass and proton numbers of an alpha particle (or cluster) and a daughter nucleus respectively. E_α is the kinetic energy of emitted alpha (cluster) particle [14]. The number of assaults per time-unit n is evaluated from the quantum-mechanical ground-state frequency in the spherical

square well:

$$n = \frac{\pi \hbar}{2\mu R^2}. \tag{5}$$

In this formalism the α-decay half-life can be expressed as follows:

$$T_{1/2} = \frac{\ln 2}{\lambda} 10^h, \tag{6}$$

where constant h (so-called *hindrance factor*) was additionally introduced for odd nuclei. The least-square fit of the radius constant r_0 was performed to known experimental half-lives of even–even α-emitters (127 cases) [14]. Only the most probable alpha decay mode at each isotope was chosen in our analysis. Obtained in this way the value of the nuclear well radius constant is equal to $r_0 = 1.23$ fm, and slightly differs from that reported in [6], as now more alpha emitters were taken and the cluster decays were not included in our analysis. Fitting procedures of *hindrance factor* were performed for odd systems with fixed r_0 value ($h = 0.25$ for odd–A and doubled for odd–odd emitters).

2.2. Results

The logarithms of the ratios of the half-lives ($T_{1/2}^{\mathrm{cal}}$), calculated using the above formalism, to the measured ones ($T_{1/2}^{\mathrm{exp}}$) for all examined nuclei are shown in figures 1-3 as a function of the neutron number N.

Large deviations from the data are observed for emitters with neutron number $N = 127$ in $_{83}$Bi - $_{92}$U elements (figure 2), where underestimations of the $\log_{10}(T_{1/2})$ reach about two orders of magnitude. Similar inaccuracies one observes also for some isotopes far from magic number $N = 126$, namely: Re ($N = 86$), Ir ($N = 99$), Bi ($N = 114$), No and Bh. Calculated half-lives of some lighter emitters ($_{52}$Te - $_{74}$W) are overestimated, but these discrepancies do not exceed one order of magnitude. The root-mean-square deviations of our estimations made for the α-decay half-lives for all considered nuclei are summarized in table 1.

3. Spontaneous fission within the WKB theory

A continuous interest in the theoretical description of the fission dynamics is observed since discovery of this phenomenon in 1938. In the present section we are going to concentrate on the spontaneous fission and its description within the WKB theory only. The results presented here will be used to understand why a simple one parameter (for even–even nuclei) model *à la Świątecki* [12], described in section 4, is able to reproduce the spontaneous fission half-lives of all known nuclei with higher accuracy than obtained using more advanced theories.

Quantum mechanically it is possible for a nucleus in its ground state to tunnel the fission barrier. The probability of the barrier penetration depends not only on the height and width of the barrier but also on the magnitude of the collective
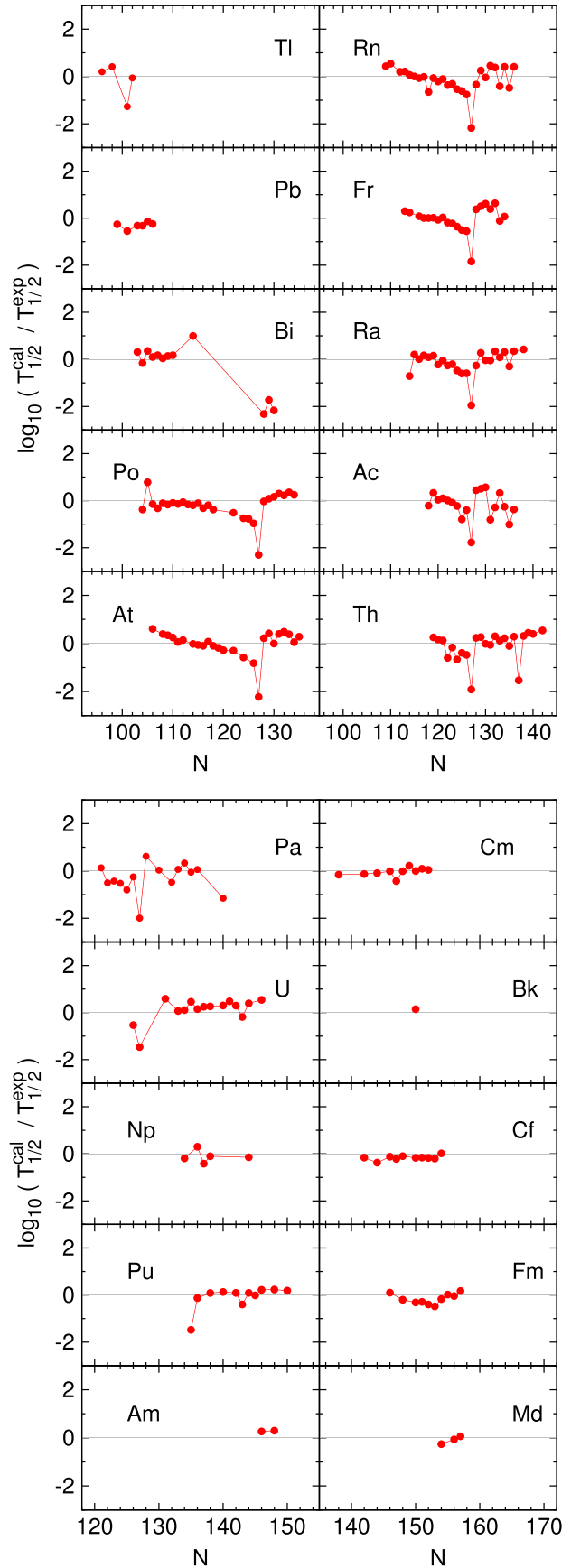
Figure 2. The same as in figure 1, but for isotopes with $81 \leqslant Z \leqslant 90$ (top) and $91 \leqslant Z \leqslant 101$ (bottom).
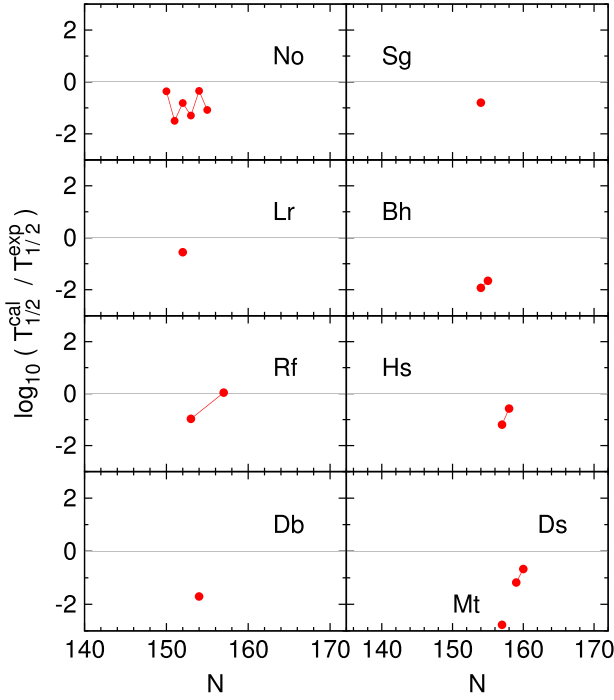
Figure 3. The same as in figure 1, but for isotopes with $102 \leqslant Z \leqslant 110$.

Table 1. Root-mean-square deviations of $\log_{10}(T_{1/2}^{\alpha})$ calculated using equation (6) with the radius constant $r_0 = 1.23$ fm.

$\pi_Z - \pi_N$	n	h	r.m.s.
e-e	127	0	0.39
e-o	97	0.25	0.66
o-e	82	0.25	0.54
o-o	54	0.5	0.79

inertia associated with the fission mode. The calculations of the spontaneous fission half-lives T_{sf} require a careful evaluation of the collective potential energy surface and the collective inertia tensor. Different models, like the macroscopic-microscopic model [15, 16] or the Hartree–Fock–Bogolubov self-consistent theories (see e.g. [17]), can be used to generate the potential energy surfaces. The collective mass parameters are usually evaluated in the cranking approximation [16, 18] or within the Generator Coordinate Method (GCM) with the generalized Gaussian Overlap Approximation (GOA) [19].

The spontaneous fission half-life is given by:

$$T_{1/2}^{\text{sf}} = \frac{\ln 2}{nP}. \qquad (7)$$

Here n is the number of assaults per time unit, associated with the frequency of zero-point vibration of a nucleus in the fission mode direction. The fission barrier penetration probability P might be evaluated within the one-dimensional WKB approximation, which leads to the following
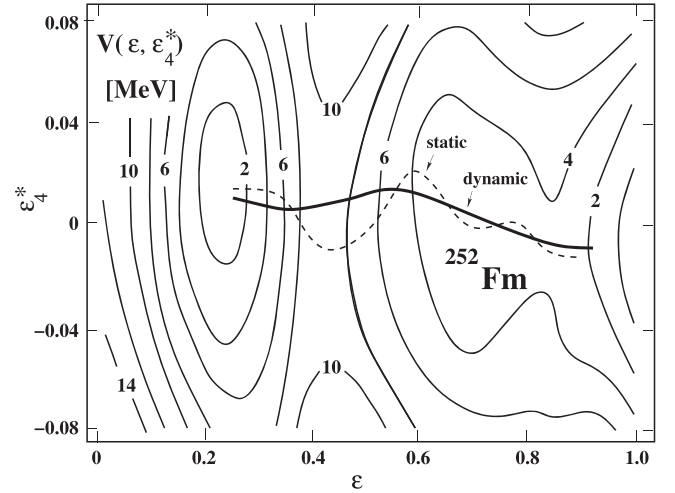
Figure 4. A contour plot of the energy surface for ^{252}Fm is shown in the two-dimensional quadrupole (ϵ) and hexadecapole (ϵ_4) deformation parameter space. The dashed line is the path of steepest descent from the saddle points to the ground state and to the scission line, the thick line is the minimal-action path. After [20].

expression:

$$P = \frac{1}{1 + \exp\{2S(L)\}}. \qquad (8)$$

The action–integral $S(L)$, calculated along a fission path $L(s)$ in the multi-dimensional space of collective coordinates is given by:

$$S(L) = \int_{s_1}^{s_2} \left\{ \frac{2}{\hbar^2} B_{\text{eff}}(s) \left[V(s) - E_{\text{gs}} \right] \right\}^{1/2} ds, \qquad (9)$$

where the integration limits s_1 and s_2, correspond to the classical turning points. E_{gs} is the energy of a nucleus in its ground state and $V(s)$ is the collective potential. $B_{\text{ss}}(s)$ is an effective inertia tensor in the multi-dimensional space of collective coordinates $\{q_i\}$:

$$B_{\text{ss}}(s) = \sum_{k,l} B_{kl} \frac{dq_k}{ds} \frac{dq_l}{ds}. \qquad (10)$$

Determination of the fission path in the deformation space plays a crucial role during evaluation of the fission probability [16]. The action integral (9) should be calculated along such a path (so-called *dynamic*), which minimizes its total value, thereby maximizing fission probability (8) [21]. In contrast to the *static path*, the dynamic one (dependent on the inertia tensor) does not have to lead through the bottom of the fission valley, as one can see in figure 4. The comparison of the fission barrier and effective inertia, corresponding to the dynamic and static fission paths for ^{250}Fm is shown in figure 5. The collective potential $V(s)$, corresponding to the dynamic path, is larger than in the case of the static one, but the dynamic inertia is smaller. It might be roughly approximated by the so-called *phenomenological inertia* fitted to the
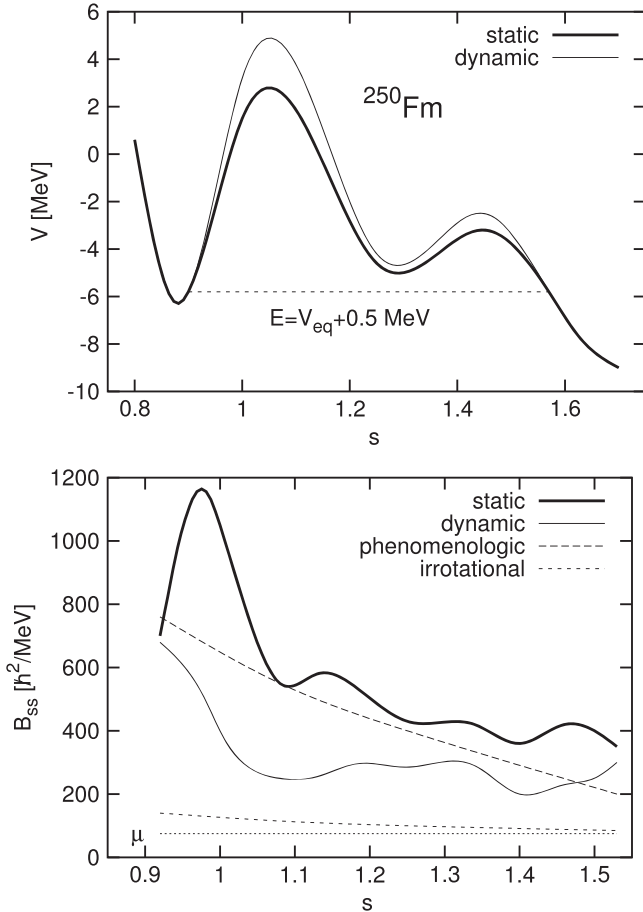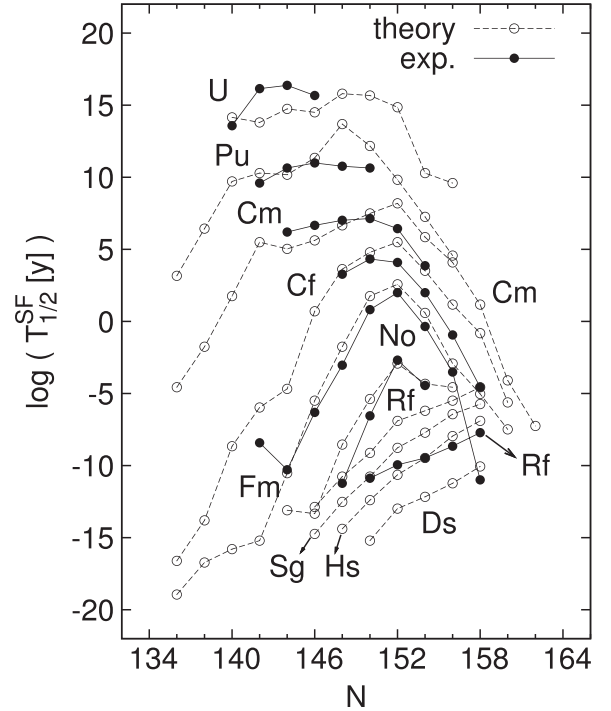
Figure 5. Collective potential (top) and inertia $B_{\rm eff}$ (bottom) along the static (thick solid line) and the dynamic (thin solid line) path to fission as a function of the relative distance between the fragment mass centres $s = R_{12}/R_0$. The reduced (μ), irrotational flow and phenomenological inertias [22] are shown for comparison. After [23].


Figure 6. Spontaneous fission half-lives of even–even transactinides and their shape isomers (II) obtained with the WKB approximation (8) along the least action (dynamical trajectory). After [21].

observed spontaneous fission half-lives [22]:

$$\mathcal{M}_{R_{12}R_{12}}^{\rm phen} = \mu\left(1 + k\frac{17}{15}\exp\left[-a\left(R_{12} - \frac{3}{4}\right)\right]\right). \quad (11)$$

Here μ is the reduced mass, corresponding to the relative distance between fragments ($R_{12} = 3/4$ for a sphere) and $a = 2.452$ is a numerical constant, determined from a fit to the exact irrotational flow inertia [24] for $k = 1$. It was found in [22] that the phenomenological inertia obtained with the constant $k = 11.5$ reproduces well the systematics of the known spontaneous fission half-lives.

Spontaneous fission half-lives of even–even transactinides evaluated in [21] are presented in figure 6. The estimates correspond to the least-action trajectories (dynamic path to fission). The WKB approximation (8) and the cranking inertia tensor were used. The two-dimensional potential energy surfaces for each isotope were obtained within the macroscopic-microscopic model using the Nilsson potential and the droplet energy for the macroscopic part. The

effects of the axial asymmetry as well as the reflection asymmetry were taken into account.

Usually, calculations of spontaneous fission half-lives are performed with an assumption, that the collective potential is equal to the Hartree–Fock–Bogolubov (HFB) binding energy or its macroscopic-microscopic approximation. The zero-point energy corrections are often ignored in papers dealing with similar problems. We shall discuss this point later. Moreover, one assumes, that the ground state is located at $\Delta E \approx 0.5$ MeV above the minimum of the HFB potential. As a justification of such a choice of ΔE one refers to experimental values of the average quadrupole phonon energy, which is about 1 MeV for actinides. Several authors even treat ΔE as a free parameter. It should be stressed, that the HFB theory, as each variational method, gives the upper limit of the ground-state energy, so the zero-point correction energy at the equilibrium point is approximately equal to the difference of the ground-state and the potential energy at the minimum (see [26]). It can be easily shown, that the zero-point energy of harmonic oscillations is equal to half of the corresponding phonon energy, i.e. ($\frac{1}{2}\hbar\omega$) [27], what is consistent with result, obtained in [28], where the coupled quadrupole and octupole vibrations in the region of Ra–Th nuclei were analyzed. This conclusion is also valid even for strongly anharmonic motion such as pairing vibrations [29].

The number n of the collective degrees of freedom depends much on the value of the zero-point energy correction. Namely, the ε_0 is proportional to $n \cdot \hbar\omega$. Thus, when the HFB equations are solved in the whole n-dimensional space of collective coordinates, all zero-point corrections should be
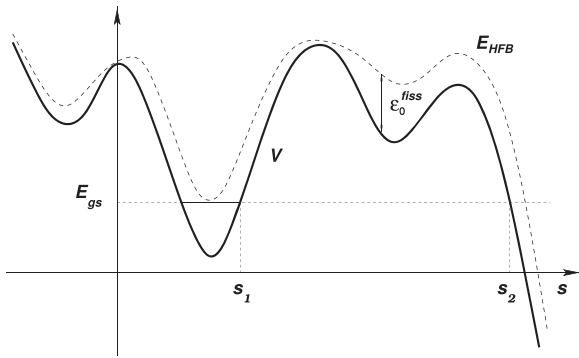
Figure 7. Collective potential $V(s)$, binding energy $E_{\text{HFB}}(s)$ and ground-state energy E_{gs} of a nucleus along the fission path s. After [23].

Figure 8. The HFB potential energy of ^{230}Th as a function of quadrupole and octupole moments. The thick solid line shows the path to cluster emission (CE) while the dashed one is drawn along the valley to asymmetric fission (AF). After [33].

substracted from the energy to obtain the collective potential:

$$V(s) = E_{\text{HFB}}(s) - \varepsilon_0^{\text{fiss}}(s), \qquad (12)$$

where $\varepsilon_0^{\text{fiss}}$ is the zero-point energy related to the fission mode (see [19]).

In the case of the one-dimensional fission barrier penetration given by the equation (9), the oscillations perpendicular to the fission path are not included during evaluation of tunnelling probability (8). In this direction the zero-point energy corrections will be approximately cancelled by the 1/2 of the corresponding phonon energy.

In figure 7 the potential and the binding energy along the fission path are plotted. As one can observe, the zero-point corrections may affect the fission barrier height.

The determination of the least action path in deformation space is not a trivial task (see [16] for further discussion). The approach, which is worth mentioning in this context, is the dynamical programming method [20, 21]. Nevertheless, applying this method one should choose the optimal distance between the mesh points to avoid large numerical errors. That is because the accuracy of calculations of the partial derivatives $\frac{\partial q_i}{\partial s}$, which depend much on the effective mass parameter (10) as well as the zero-point energy correction (see e.g. [19]) along the fission path, strongly depends on the grid size. The von Ritz implementation (see appendix of [30]) might be useful as an alternative for practical calculations of the spontaneous fission probability. The main advantage of this method is that the influence of the zero-point energy correction along the dynamical paths might easily be taken into account.

Typically the mass yield of the spontaneous fission fragments of the actinide nuclei is spread between 70 and 180. Usually, except for the bimodal fission of ^{258}Fm and neighbouring nuclei, the asymmetric fission is more probable than the symmetric one and the most populated mass of the heavier fragment is around 140. A very mass asymmetric spontaneous fission process, when the mass of the lighter fragment is around 20, is already considered as a cluster emission. This type of radioactivity was predicted in 1980 by Sandulescu and co-workers [31] and four years later it was discovered by Rose and Jones [32], who observed the spontaneous emission
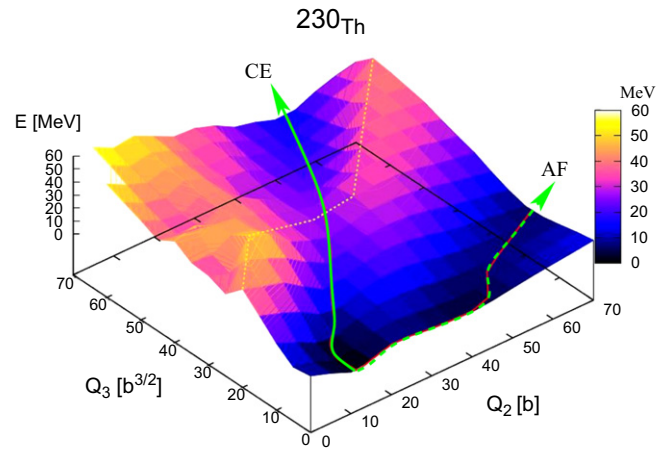
of ^{14}C from ^{223}Ra. The cluster emission is a very rare process. Its relative branching ratio to the α-decay is of the order 10^{-10} to 10^{-17}. Nevertheless, in the last two decades one has observed clusters from ^{14}C to ^{34}Si emitted by actinide nuclei from ^{221}Fr to ^{242}Cm. In all cases the residual nucleus was always close to the double magic ^{208}Pb.

In general, there are two alternative theoretical descriptions of the spontaneous cluster decay. One can use a fission-like mechanism to reproduce the main features of the process [31, 33–36] or one assumes, in close analogy to the α-decay, that a cluster, performed by a nonadiabatic mechanism inside nucleus, penetrates the potential barrier created by the Coulomb and nuclear interaction with a daughter nucleus [37–40]. The potential energy landscape of ^{230}Th obtained by the HFB calculation with the Gogny D1S force [33] is presented in figure 8 as a function of quadrupole (Q_2) and octupole (Q_3) moments. Two valleys leading to the cluster emission (CE, solid line) and to the asymmetric fission (AF, dashed line) are visible in the plot.

The above theoretical models, which describe the cluster-radioactivity, are rather complex and contain adjustable parameters. Last year we showed in [6], that within a simple Gamow model with only one adjustable parameter (radius constant) common for the α-decay and the cluster emission one can describe with good accuracy the experimental systematics of half-lives for the cluster radioactivity of even–even nuclei. An additional hindrance constant was introduced in [6] to describe the cluster emission probability from odd–even, even–odd and odd–odd isotopes.

4. Simple phenomenological formula for the spontaneous fission half-lives

Encouraged by the good result for $T_{1/2}^{\alpha}$ obtained in the Gamow theory, we are going in the following to describe in another simple model the spontaneous fission half-lives of

transactinide nuclei. Namely, we shall adopt the Świątecki idea from 1955 [12], who found a simple relation between the spontaneous fission half-lives and the experimental mass deviations from their liquid drop estimates. The crucial ingredient of the model is the liquid drop formula which is described in the next subsection.

4.1. The liquid drop model

In our analysis we decided to use the modern version of the macroscopic-microscopic model [13]. The macroscopic part of this Lublin–Strasbourg Drop (LSD) mass formula (in MeV units) is as follows:

$$
\begin{aligned}
M_{\text{LSD}}&(Z, N, \text{def}) \\
&= 7.289034 \cdot Z + 8.071431 \cdot N - 0.00001433 \cdot Z^{2.39} \\
&\quad - 15.4920\left(1 - 1.8601 I^2\right)A \\
&\quad + 16.9707\left(1 - 2.2938 I^2\right)A^{2/3} \, B_{\text{surf}}(\text{def}) \\
&\quad + 3.8602\left(1 + 2.3764 I^2\right)A^{1/3} B_{\text{cur}}(\text{def}) \\
&\quad + 0.70978 \, Z^2/A^{1/3} B_{\text{Coul}}(\text{def}) - 0.9181 \, Z^2/A \\
&\quad - 10 \exp\left(-4.2 \, |I|\right) B_{\text{cong}}(\text{def}) + E_{\text{o-e}}.
\end{aligned}
\tag{13}
$$

where the odd–even energy term $E_{\text{e-o}}$ is given in [41]. In equation (13) $A = Z + N$ denotes the mass number, $I = (N - Z)/A$ reduced isospin and B_{surf}, B_{cur}, B_{Coul} and B_{cong} are relative to the sphere: surface, curvature, Coulomb and congruence (see [41]) energies. The parameters in the first and the last row in Equation (13) are taken from [42], while the remaining eight parameters were fitted to the data.

4.2. Simple model à la Świątecki

Following the idea presented in [12], we are going to find an approximate functional dependence of the logarithms of spontaneous fission half-lives, corrected by mass shifts $k\delta M$, on proton Z and neutron N numbers:

$$
f(Z, N) = \log_{10}\left[T_{1/2}^{\text{sf}}/y\right] + k\delta M(Z, N).
\tag{14}
$$

The δM is a ground-state microscopic energy, defined as:

$$
\delta M_{micr}^{\exp}(Z, N) = M_{\exp}(Z, N) - M_{\text{LSD}}(Z, N, 0),
\tag{15}
$$

where $M_{\exp}(Z, N)$ is a measured mass of isotope, taken from [43] and $M_{\text{LSD}}(Z, N, 0)$ were calculated using formula (13).

Fitting procedures included 35 fissioning even–even isotopes ($Z \leqslant 102$) with measured masses and half-lives. The smooth dependence for even–even isotopes was achieved for factor $k = 7.7 \, \text{MeV}^{-1}$, as is shown in figure 9. The curves, fitted for odd–A and odd–odd nuclei are shifted by a constant-*hindrance factor*. It is equal to $h = 2.5$ for odd–A isotopes and doubled for odd–odd systems. Formula for spontaneous
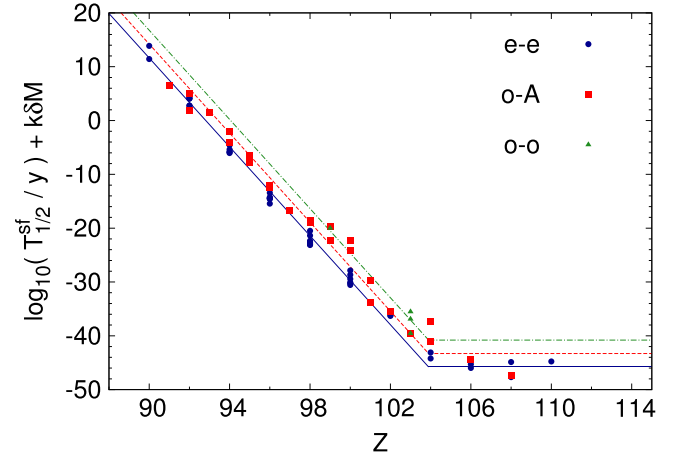
Figure 9. Logarithms of the observed spontaneous fission half-lives corrected with masses "shifts" $k\delta M$ as a function of proton number.
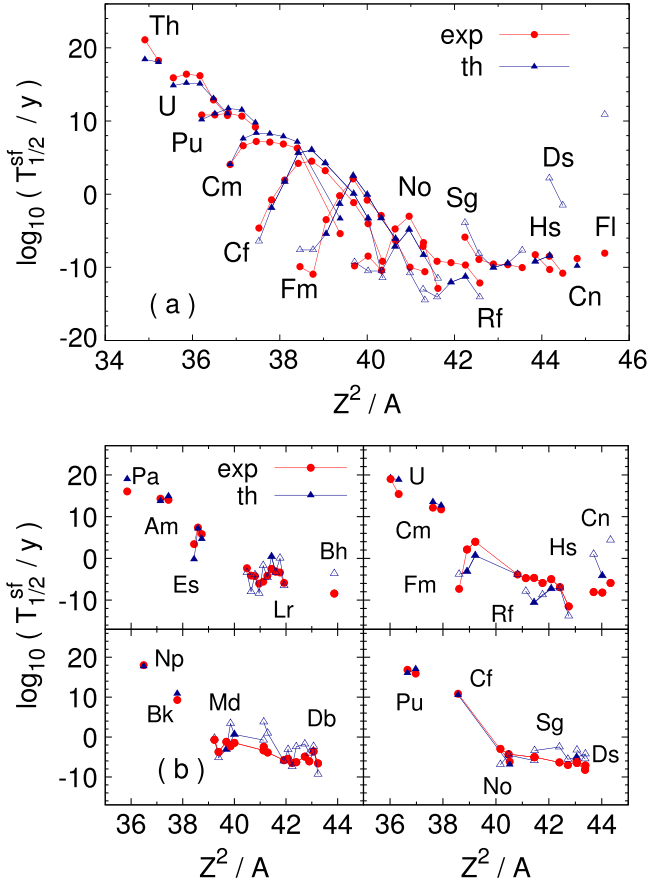
fission half-lives in years is given by:

$$
\log_{10}\left(T_{1/2}^{\text{sf}}/y\right) = -4.1 \cdot \min(Z, 103) + 380.2
$$
$$
- 7.7 \, \delta M(Z, N) +
\begin{cases}
0 & \text{for even} - \text{even}, \\
2.5 & \text{for odd} - A, \\
5 & \text{for odd} - \text{odd}.
\end{cases}
\tag{16}
$$

Spontaneous fission half-lives, calculated using formula equation (16) are presented and compared with the data in figure 10. Isotopes which for masses are still not measured are marked with open symbols. In these cases the empirical mass corrections, given by equation (15) were replaced by the theoretical ones, taken from [42]. A very good accuracy was achieved in the actinides region, where the root-mean-square deviation of $\log T_{1/2}^{\text{sf}}$ is equal to 1.18 and it grows to 1.57 when odd–A and odd–odd nuclei are taken into account. The r.m.s. deviation reaches 1.96, when one includes in the analysis the super-heavy isotopes, for which the liquid drop barrier vanishes. The quality of reproduction of known lifetimes by this simple model is even better than that presented in figure 6 obtained in the macroscopic-microscopic model with the cranking inertia [21]. Also more advanced modern calculations based on the HFB theory with the Skyrme (see e.g. [44, 45] or Gogny force [46] did not bring better estimates for large sample of isotopes than the simple formula presented above. It raises a question: why does the simple Świątecki model work? To understand it, one has to recall Świątecki's topographical theorem and the results obtained using dynamic (least action) trajectories to fission described in section 3.

4.3. The topographical theorem

It was shown in [13, 30, 47] that the LSD model (13), whose parameters were fitted to the experimental ground-state masses only, is able to reproduce well the fission barrier heights of light, medium and heavy nuclei, when the microscopic part of the ground-state binding energy is taken into account. According to the topographical theorem, proposed by Myers and Świątecki [41], the mass of a nucleus in a

Figure 11. Experimental fission barrier heights compared with their estimates evaluated according to the topographical theorem of Świątecki and the LSD macroscopic energy. After [49].

Figure 10. Spontaneous fission half-lives of even–even (a) and odd (b) nuclei, calculated using formula (16) (triangles) in comparison to the experimental values [14] (dots).

Figure 12. Schematic plot of a fission barrier in the form of two inverted parabolas smoothly joined at the scission point.

saddle-point is mostly determined by the macroscopic part of its binding energy. The influence of the shell effects on the saddle-point energy is rather weak and in the first approximation might be ignored. So, one can evaluate the fission barrier height as a difference between the macroscopic (here LSD) and the experimental mass:
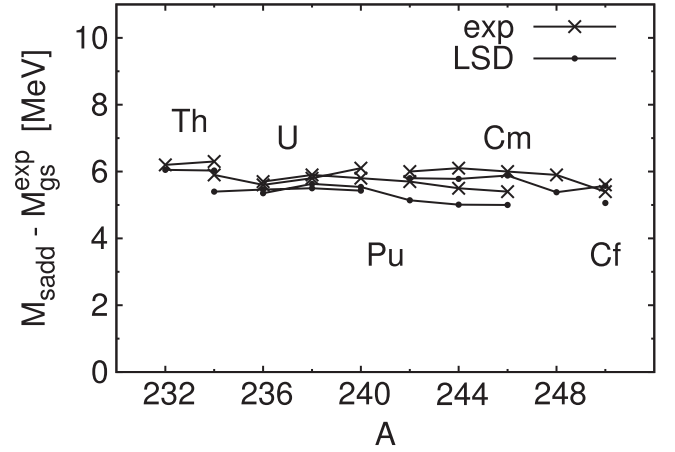
$$V_B(Z, N) = M_{\mathrm{LSD}}(Z, N, \mathrm{saddle}) - M_{\mathrm{exp}}^{g.s.}, \qquad (17)$$

where $M_{\mathrm{LSD}}(Z, N, \mathrm{saddle})$ is a macroscopic part of LSD mass formula (13) taken in a saddle point.
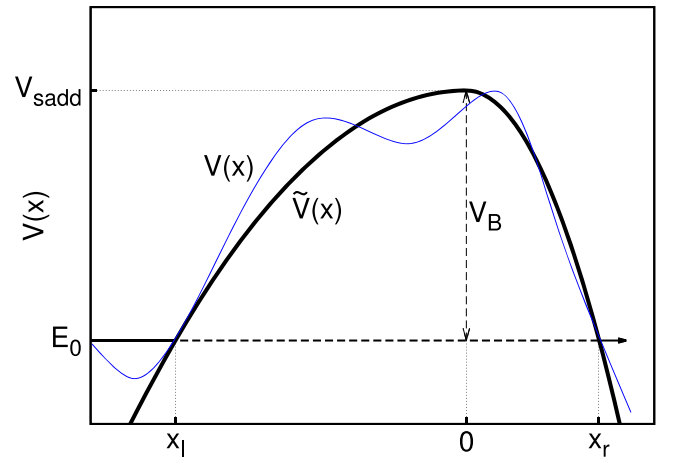
The comparison of the experimental fission barrier height of actinide nuclei with their estimates, evaluated according to the topographical theorem of Świątecki using the LSD macroscopic energy, is presented in figure 11. The r.m.s. deviation of the estimates from the data for all 18 considered fission barriers' heights is 0.31 MeV only, which proves that the topographical theorem really works [49]. All estimates do not exceed 0.67 MeV and lie below the experimental values (except ^{250}Cf), which give a place for not washed out shell effects around the saddle point energy.

4.4. Justification of the simple formula for T_{sf}

Let us consider a one-dimensional fission barrier along the least-action (dynamic) trajectory described in section 3.

Assuming, that the fission path is parameterised by the collective coordinate s, one can evaluate the potential $V(s)$ and the mass parameter $B_{ss}(s)$ corresponding to this path. The fluctuations of the inertia B_{ss} along the least-action trajectory becomes smaller and smaller, when one increases the number of collective coordinates. In particular, the collective pairing degrees of freedom significantly wash out the fluctuations of the inertia function [50].

A simple transformation:

$$x(s) = \int_{s_{\mathrm{sadd}}}^{s} \sqrt{\frac{B_s s(s')}{m}} \, \mathrm{d}s', \qquad (18)$$

brings us to the new coordinate $x(s)$, in which the inertia becomes constant $B_{xx} = m$. The lower integration limit in equation (18) is chosen at point $x = 0$, corresponding to the saddle point s_{sadd}. The collective potential in the new coordinate, schematically plotted in figure 12, can be approximated by two inverted parabolas of different stiffnesses

C_l and C_r having the maximum at the saddle point:

$$\widetilde{V}(x) = \begin{cases} V_{\text{sadd}} - \dfrac{1}{2}C_l\, x^2 & \text{for } x < 0, \\ V_{\text{sadd}} - \dfrac{1}{2}C_r\, x^2 & \text{for } x > 0, \end{cases} \qquad (19)$$

The stiffnesses of the $\widetilde{V}$ potential are chosen in such a way that the action-integral S (9) becomes equal:

$$\int_{-x_l'}^{x_r'} \sqrt{\frac{2\,m}{\hbar^2}\left[V(x) - E_0\right]}\,\mathrm{d}x$$
$$= \int_{-x_l}^{x_r} \sqrt{\frac{2\,m}{\hbar^2}\left[\widetilde{V}(x) - E_0\right]}\,\mathrm{d}x \qquad (20)$$

where pairs $(-x_l', x_r')$ and $(-x_l, x_r)$ are classical left and right turning points for the true and approximative potential respectively. The last integral in equation (20) can be rewritten as

$$S = \sqrt{\frac{2\,m}{\hbar^2}}\left\{ \int_0^{x_l} \sqrt{V_B - \frac{C_l x^2}{2}}\,\mathrm{d}x + \int_0^{x_r} \sqrt{V_B - \frac{C_r x^2}{2}}\,\mathrm{d}x \right\},$$

where $V_B = V_{\text{sadd}} - E_0$ is the fission barrier height. After a little algebra the action integral becomes

$$S = \frac{\pi}{2\hbar}V_B\left(\sqrt{\frac{m}{C_l}} + \sqrt{\frac{m}{C_r}}\right) = \frac{\pi}{\hbar}V_B\frac{\omega_l + \omega_r}{2\omega_l\omega_r}, \qquad (21)$$

where $\omega_l = \sqrt{C_l/m}$ and $\omega_r = \sqrt{C_r/m}$ are frequencies of the left and right (inverted) oscillator respectively. Introducing the average oscillator frequency

$$\widetilde{\omega} = \frac{2\omega_l\omega_r}{\omega_l + \omega_r}, \qquad (22)$$

one can bring the action integral to the following form:

$$S = \frac{2V_B}{\hbar\widetilde{\omega}}. \qquad (23)$$

In our approximation the action integral is proportional to the fission barrier height measured in the energy quanta of the harmonic oscillator, which approximates the fission barrier form. For the action-integral $S > 1$ the logarithm of the spontaneous fission half-lives (7) can written as

$$\log\left(T_{1/2}^{\text{sf}}/\text{y}\right) = 2\,S - \log(n) - 0.3665 \qquad (24)$$

where the constant $0.3665 = \log[\ln(2)]$ and n is the number of assaults of nucleus on the fission barrier per year. Having in mind that according to the topographical theorem the fission barrier height is $V_B = M_{\text{LSD}}^{\text{sadd}} - M_{\text{exp}}^{gs}$ and making use of the relation (23), one can rewrite the last equation as follows:

$$\log\left(T_{1/2}^{\text{sf}}/\text{y}\right) + \frac{4\delta M_{\text{micr}}^{\text{exp}}}{\hbar\widetilde{\omega}} = \frac{4V_B^{\text{LSD}}}{\hbar\widetilde{\omega}} - \log(n) - 0.3665, \quad(25)$$

where $\delta M_{\text{micr}}^{\text{exp}}$ was defined in equation (15) and $V_B^{\text{LSD}} = M_{\text{LSD}}^{\text{sadd}} - M_{\text{LSD}}^{\text{sph}}$ is the liquid drop fission barrier height. The right hand side of equation (25) is a very smooth function of nucleon numbers as it is defined only by global
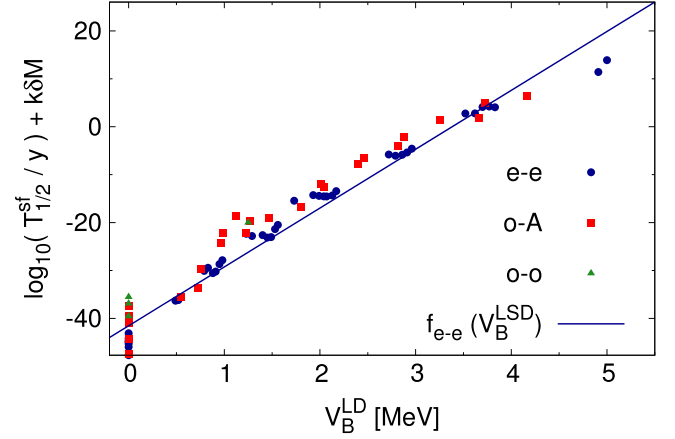
Figure 13. Logarithms of the observed spontaneous fission half-lives corrected with $7.7\delta M$ as a function of liquid drop barrier height.

properties of the nucleus. Note, that the derived equation has the same structure as the phenomenological formula of Świątecki [14], which proves his Ansatz.

The liquid drop barrier height of actinides decreases almost linearly in function of Z from 4.3 MeV for $Z = 90$ to 0 for $Z \geqslant 103$ [47]. The fission barrier of finite height appears in the super-heavy nuclei mostly due to the shell effects in the ground state. The smooth dependence of logarithms of spontaneous fission half-lives, corrected by the ground state shell-plus-pairing effects, on the LSD fission barrier heights, shown in figure 13 for even–even (e–e), odd A (o–A) and odd–odd (o–o) nuclei. The data for e–e isotopes lie very close to the straight line, which validates Equation (25). The data for o–A and o–o are above this line, which is due to the specialization energy, which increases the fission barrier heights. As was mentioned before, stability of super-heavy nuclei is determined by shell effects. The liquid drop barrier of these isotopes vanishes, which is visible in figure 13 for nuclei with $Z \geqslant 103$.

5. Conclusions

The following conclusions can be drawn from our investigation:

- A simple, consistent model was applied to reproduce alpha-decay half-lives in 360 nuclei with atomic number $52 \leqslant Z \leqslant 110$.
- The model reproduces α-decay half-lives with quite good accuracy; the root-mean-square deviation of $\log_{10}(T_{1/2}^\alpha)$ for even–even isotopes is equal to 0.39.
- Large underestimations in half-lives of $N = 127$ isotopes arises from strong shell effects, which are not considered in this simple model.
- Semi-empirical formula for the spontaneous fission half-lives, depending on proton number and the ground state microscopic corrections, reproduces data for even–even super-heavy nuclei with reasonable accuracy.

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- Quality of spontaneous fission half-lives evaluation breaks down for nuclei with not measured yet masses.
- The logarithms of the spontaneous fission half-lives, corrected by the ground state shell-plus-pairing effects, are roughly proportional to the macroscopic barrier heights in nuclei up to Z = 102.

Acknowledgments

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