Transient dynamics of a quantum dot embedded between two superconducting leads and a metallic reservoir

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We study time-dependent subgap properties of a quantum dot (QD) embedded between two superconductors and another metallic lead, solving the Heisenberg equations of motion by the Laplace transform technique subject to the initial conditions. Focusing on the response of the system induced by a sudden coupling of the QD to external reservoirs, we analyze the transient currents and their differential conductance. We also derive analytical expressions for measurable quantities and find that they oscillate in time with the frequency governed by the QD coupling to superconducting reservoirs. Such quantum oscillations are damped due to relaxation processes caused by the normal lead, whereas their period is controlled by the phase difference $\phi$ between the order parameters of superconducting leads (except the case $\phi = \pi$, when all observables evolve to their stationary values without any oscillations). We also explore time-dependent development of the subgap quasiparticles and find their signatures measurable in the differential conductance. We evaluate (numerically and analytically) three typical time scales, characterizing the initial and large-time stages of the transient dynamics which in the asymptotic limit ($t \to \infty$) drives these subgap quasiparticles to the true Andreev states.

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I. INTRODUCTION

Transient effects of the quantum dot (QD) systems have been intensively studied over recent years, providing useful insight into the electron transport properties. These effects could be of special importance in experiments on nanoscopic devices, where different types of time-dependent pulses can effectively control the electron flow. Transient effects have been studied, both theoretically and experimentally, for the QDs coupled to the metallic (conducting) electrodes [1–67] and in the presence of superconducting reservoirs [68–84]. Numerous theoretical approaches have been developed to deal with such time-dependent problems, e.g., the iterative influence-functional path integral [35], Keldysh formalism and time-dependent partition-free approach [40], weak-coupling continuous-time Monte-Carlo method [27], and many other techniques [53,61].

The coherent oscillations and current beats have been found in a short-time response of a system upon abrupt change of the bias voltage [9,14]. From the periods of the current beats it is possible to estimate the values of the QDs energy levels or the hopping parameters between them [38,51,57]. The transient current characteristics can also be used to determine the spin relaxation time in some QD systems [4]. Such phenomena have been investigated for QDs coupled to the normal leads as a result of the bias voltage pulse [5,22,29,31,36,41,53,60,75], driven by an arbitrary time-dependent bias [26,27,40,53,61], by a sequence of rectangular pulses applied to the input lead [17,32] or applied to the contact gradually switched on in time [25]. The transient dynamics has also been studied for QD after a sudden symmetrical connection to the leads [27,37,78,85] or asymmetrically coupled to electrodes following a sudden change of the QD energy levels [11]. The transient heat generation driven by a steplike pulse bias within the Anderson-Holstein model or the time-dependent current through QD suddenly coupled to a vibrational mode have been studied in nanostructures with the normal [19,30,47,56,63] or superconducting electrodes [71].

Technological progress in the real-time detection of single electrons has opened a possibility for studying electron transport from a perspective of the stochastic processes. Among theoretical tools for investigating the electron hopping statistics there are, e.g., the full counting statistics (FCS) and the waiting time distribution (WTD) [54,55,62,66,73,79]. These theoretical techniques have been successfully applied to investigations of the transient processes via QD coupled to the normal leads [62] or in hybrid systems with superconductors [66,73,79]. Time-dependent processes are often investigated numerically, however, in exceptional cases some analytical results can give deeper insight into the considered problem. For instance, WTD in the normal lead-QD-superconducting junction exhibit the coherent oscillations between the empty and doubly occupied QD [73]. Similarly, some analytical calculations are possible for the energy transport in the polaronic regime described within the FCS method [59], for transient dynamics after a quench [64], for a phononic heat transport in the transient regime [65], or for transient heat generation under a steplike bias pulse [44].

In this paper we analyze the subgap transport properties of a system comprised of a single QD which is tunnel coupled to one metallic (normal) and two superconducting electrodes, focusing on transient effects driven by abrupt coupling of these constituents. It is natural that oscillations of the transient current would appear as a result of such quench, and they should depend on initial conditions of the system. Such hybrid nanostructures with QD between the normal and supercon-
...ducting electrodes reveal many interesting effects with potential applications in nanoelectronics, spintronics, or quantum computing [29,30,42,63,64]. The superconducting reservoir affects the QD via proximity effect and could be responsible for the Cooper-pair tunneling and Josephson currents, even in the absence of any bias voltages. An additional normal electrode coupled to the system allows for good control of the electron transport [86–92] and could significantly affect the transient phenomena. Our goal is to investigate analytically the time-dependent QD occupation, the currents flowing from the normal and superconducting leads, the induced QD pairing, the conductance, and the time evolution of the Andreev bound states (ABS) [90,93–97]. The formation of ABS signifies that superconducting correlations are induced in the QD via the proximity effect. We investigate the appearance in time of these states and study their spin dependence. To perform analytical time-dependent calculations we assume that the superconducting gap of both superconducting leads is the largest energy scale and we put it equal to infinity. Nevertheless, the realistic physics in the Andreev transport regime is still captured in this limit. Knowledge of the analytical formulas allows us to find the answers to such questions as: (i) how do the considered quantities and their characteristics depend on the QD energy levels or the individual coupling of the QD with a given lead, (ii) what is the time period and frequency of these time-dependent quantities, and many related issues. Our investigations allow us also to analyze time evolution of the subgap quasiparticles and their dependence on the phase difference between the superconducting reservoirs. In our calculations we apply the equation of motion method for the second quantization operators and obtain their analytical form using the Laplace transform technique. Numerical calculations could provide results only for a specific choice of parameters and would not give deep insight into specific dependence of here considered quantities of our system. In this context the analytical calculations are much more general and could have some advantage over numerical data.

The paper is organized as follows. In Sec. II we present our model and discuss the theoretical formalism. The time-dependent QD occupancy is analyzed in Sec. III, whereas Sec. IV is devoted to the proximity-induced pairing effects. The normal and superconducting transient currents through the QD are analyzed in Sec. V and in Sec. VI we discuss the subgap conductance. In Sec. VII we briefly address the correlation effects and finally, in Sec. VIII, we summarize our study.

II. MODEL AND THEORETICAL DESCRIPTION

The system under consideration consists of a QD placed between two superconducting leads (S1 and S2) and one metallic electrode N, see Fig. 1.

The model Hamiltonian for this system can be written in the following form: \( H = H_{S1} + H_{S2} + H_N + H_{QD} + H_{int} \), where \( H_{Sj} \) (\( j = 1,2 \)) describes electrons in the left or right superconducting lead

\[
H_{Sj} = \sum_{q\sigma} \epsilon_{q\sigma} c_{q\sigma}^+ c_{q\sigma} + \sum_{q_i} (\Delta_j c_{-q_i,\downarrow}^+ c_{q_i,\downarrow}^+ + \text{H.c.}),
\]

\( H_N \) refers to the normal lead, \( H_N = \sum_{k\sigma} \epsilon_{k\sigma} c_{k\sigma}^+ c_{k\sigma} \). \( H_{QD} \) describes the QD, \( H_{QD} = \sum_{\sigma} \epsilon_{\sigma} c_{\sigma}^+ c_{\sigma} + \text{H.c.} \). Electron transitions between external leads and the QD are established by the tunnel Hamiltonian:

\[
H_{int} = \sum_{k,\sigma} V_{k\sigma} c_{k\sigma}^+ c_{\sigma} + \sum_{j=1,2} \sum_{q\sigma} V_{q\sigma} c_{q\sigma}^+ c_{\sigma} + \text{H.c.} \]  

where \( \Delta_j \), of the superconducting leads to be phase dependent, \( \Delta_j = |\Delta_j| \exp(i\phi_j) \). In our notation \( k \) \( (q_i) \) shall denote itinerant states of the normal (superconducting) lead.

We are going to study time response of this system on abrupt switching of the coupling parameters. We shall thus calculate the time-dependent QD occupations, \( n_\sigma(t) \), and the currents flowing from the leads, \( j_{k\sigma}(t), j_{q\sigma}(t) \). Additionally we will compute \( \langle c_{\sigma}(t) c_{\sigma}(t) \rangle \), which corresponds to the electron pairing induced at QD via proximity effect. In what follows we assume that all couplings between the QD and the leads are suddenly switched on at \( t = 0^+ \) (for \( t \leq 0 \) the QD is decoupled from the leads). The time evolution of the considered quantities for \( t > 0 \) depends on the initial QD filling and the chemical potentials. As time goes to infinity, we reproduce the stationary limit results known for the corresponding system. In this paper we use the Laplace transform method and our strategy in the calculations is as follows: We construct the closed set of the equation of motion for creation and annihilation operators (in the Heisenberg representation) \( c_\sigma(t), c_{k\sigma}(t), c_{q\sigma}(t) \), \( c_\sigma^+(t), c_{k\sigma}^+(t), c_{q\sigma}^+(t) \), \( c_{q\sigma}^+(t) \), using the Laplace transformations for these differential equations we obtain the set of coupled algebraic forms \( c(s) = \int_0^\infty dt e^{-st} c(t) \) for all considered operators. For instance, the QD occupation \( n_\sigma(t) \) can be found from the relation

\[
n_\sigma(t) = \langle L^{-1}[c_\sigma^+(s)](s) \rangle \cdot L^{-1}[c_\sigma(s)(t)],
\]

where \( L^{-1}[a(s)](t) \) stands for the inverse Laplace transform of \( a(s) \) and \( \langle \ldots \rangle \) is the quantum statistical averaging.

Let us find the Laplace transforms of operators \( c_{\sigma}(t) \) and \( c_{q\sigma}(t) \) which are required to calculate the QD occupancy \( \langle c_{\sigma}(t) c_{\sigma}(t) \rangle = n_\sigma(t) \), the QD induced pairing \( \langle c_\sigma(t) c_\sigma(t) \rangle \), and the currents flowing from the leads. We write the Laplace transformed equations of motions for the closed set of twelve...
 operators (in the Heisenberg representation): $c_1$, $c_1^\dagger$, $c_k$, $c_k^\dagger$, $c_{q_j}$, $c_{q_j}^\dagger$, $c_{q_j \downarrow}$, $c_{q_j \uparrow}$, $j = 1, 2$.

\[
\begin{align*}
(s + i\epsilon) c_1(t) &= -i \sum_{r=k,q_j} V_r c_{r\uparrow}(t) + c_1(t), \quad (4a) \\
(s + i\epsilon) c_{q_j\downarrow}(t) &= -iV_{q_j} c_{q_j\downarrow}(t) + c_{q_j\downarrow}(t), \quad (4b) \\
(s + i\epsilon) c_{q_j\uparrow}(t) &= -iV_{q_j} c_{q_j\uparrow}(t) + c_{q_j\uparrow}(t), \quad (4c) \\
(s - i\epsilon) c_1^\dagger(t) &= i \sum_{r=k,q_j} V_r c_{r\uparrow}^\dagger(t) + c_1^\dagger(t), \quad (5a) \\
(s - i\epsilon) c_{q_j\downarrow}^\dagger(t) &= iV_{q_j} c_{q_j\downarrow}^\dagger(t) - i\Delta_j c_{q_j\uparrow}(t) + c_{q_j\downarrow}^\dagger(t), \quad (5b) \\
(s - i\epsilon) c_{q_j\uparrow}^\dagger(t) &= iV_{q_j} c_{q_j\uparrow}^\dagger(t) - i\Delta_j c_{q_j\downarrow}(t) + c_{q_j\uparrow}^\dagger(t), \quad (5c) \\
(s - i\epsilon) c_{k\downarrow}^\dagger(t) &= iV_k c_{k\uparrow}^\dagger(t) + c_{k\downarrow}^\dagger(t). \quad (5d)
\end{align*}
\]

From Eqs. (4a)–(4d) and Eqs. (5a)–(5d) we get

\[
\begin{align*}
c_1(t) M_1^{(+)}(s) &= A(s) - iK(s) c_1^\dagger(s), \quad (6a) \\
c_1^\dagger(t) M_1^{(-)}(s) &= B(s) - iK^*(s) c_1(t), \quad (6b)
\end{align*}
\]

where

\[
K(s) = \sum_{j=1,2} \frac{V_{q_j}^2 \Delta_j}{s^2 + \epsilon_{q_j}^2 + |\Delta_j|^2}.
\]

\[
A(s) = -\sum_{j=1,2} \frac{V_{q_j} \left(\Delta_j c_{q_j\uparrow}(0) + i(s - i\epsilon_{q_j}) c_{q_j\downarrow}(0)\right)}{s^2 + \epsilon_{q_j}^2 + |\Delta_j|^2} - i \sum_k V_k c_{k\uparrow}(0) + c_1(0),
\]

\[
B(s) = \sum_{j=1,2} \frac{V_{q_j} \left(\Delta_j^* c_{q_j\downarrow}(0) + i(s + i\epsilon_{q_j}) c_{q_j\uparrow}(0)\right)}{s^2 + \epsilon_{q_j}^2 + |\Delta_j|^2} + i \sum_k V_k c_{k\uparrow}(0) + c_1^\dagger(0),
\]

\[
M_1^{(+/-)}(s) = s \pm i\epsilon_{q_j} + \sum_{j=1,2} \frac{V_{q_j}^2 (s \mp i\epsilon_{q_j})}{s^2 + \epsilon_{q_j}^2 + |\Delta_j|^2} + \sum_k \frac{V_k^2}{s \pm i\epsilon_k}.
\]

Solving Eqs. (6a), (6b) we obtain for $c_1(t)$

\[
c_1(t) = \frac{M_1^{(-)}(s) A(s) - iK(s) B(s)}{M_1^{(+)}(s) M_1^{(-)}(s) + K(s) K^*(s)}. \quad (11)
\]

Repeating the same procedure to the set of operators: $c_1$, $c_1^\dagger$, $c_k$, $c_k^\dagger$, $c_{q_j\downarrow}$, $c_{q_j\uparrow}$, $c_{q_j\uparrow}$, $c_{q_j\uparrow}$, and $c_{q_j\downarrow}$, one can get

\[
c_1(s) = \frac{M_1^{(-)}(s) B^+(s) + iK(s) A^+(s)}{M_1^{(+)}(s) M_1^{(-)}(s) + K(s) K^*(s)}. \quad (12)
\]

Laplace transforms of $c_1$ and $c_1^\dagger$ can be obtained, taking the hermitian conjugation of $c_1$ and $c_1^\dagger$, respectively.

In the wide-band limit approximation and for $|\Delta_j| = \infty$ the functions $M_j^{(+/-)}(s)$ and $K(s)$ can be expressed in the following analytical forms: $M_j^{(+/-)}(s) = s \pm i\epsilon_{q_j} + \Gamma_j/2$, and $K(s) = (\Gamma_{SL} e^{\delta q_1} + \Gamma_{SR} e^{\delta q_2})/2$. Here we have assumed $\Gamma_j/2 = 2\pi \sum_{k,q_j} V_{k,q_j}^2 \delta(e - \epsilon_{k,q_j})$ and $\epsilon_{k,q_j} = \epsilon_k$, $\epsilon_{q_j\downarrow} = \epsilon_{q_j\uparrow} = \epsilon_{q_j}$.

As an example, let us present the explicit form of the Laplace transform for $c_1(t)$

\[
c_1(s) = \frac{1}{(s - s_3)(s - s_4)} \left\{ (s - i\epsilon_1 + \Gamma_N/2) c_1(0) - i \sum_k V_k c_{k\uparrow}(0) \right. \\
\left. - \sum_{j=1,2} \frac{iV_{q_j}(s - i\epsilon_{q_j}) c_{q_j\downarrow}(0) + V_{q_j} \Delta_j c_{q_j\uparrow}(0)}{s^2 + \epsilon_{q_j}^2 + |\Delta_j|^2} \right\}.
\]

\[
\frac{- \frac{i}{2} (\Gamma_{SL} e^{\delta q_1} + \Gamma_{SR} e^{\delta q_2}) \left[ c_1^\dagger(0) + \sum_k V_k c_k^\dagger(0) \right] + \sum_{j=1,2} \frac{iV_{q_j}(s + i\epsilon_{q_j}) c_{q_j\downarrow}(0) + V_{q_j} \Delta_j c_{q_j\uparrow}(0)}{s^2 + \epsilon_{q_j}^2 + |\Delta_j|^2} \right\}.
\]

where $s_{3,4} = \frac{1}{2} [-i(\epsilon_1 - \epsilon_{q_j} - \Gamma_N \pm \delta \sqrt{2}], \delta = (\epsilon_1 + \epsilon_{q_j})^2 + \Gamma_{12},$ and $\Gamma_{12} = \Gamma_{SL}^2 + \Gamma_{SR}^2 + 2\Gamma_{SL} \Gamma_{SR} \cos(\delta q_1 - \delta q_2)$.

Note that in the formula (13) there appears the finite superconducting energy gap $\Delta_j$. The limit $|\Delta_j| = \infty$ will be imposed later on, when computing the expectation values of the product of two corresponding operators, e.g., $\langle c_1(t) c_1^\dagger(t) \rangle$ or $\langle c_{q_j\uparrow}(t) c_{q_j\downarrow}(t) \rangle$. Additionally, expression for $c_{q_j\sigma}(s)$ needed for calculations of the currents flowing between the QD and the superconducting leads can be obtained from Eqs. (4b), (4c), (11), (12) and it reads

\[
c_{q_j\sigma}(s) = \frac{1}{s^2 + \epsilon_{q_j}^2 + |\Delta_j|^2} \left[ (s - i\epsilon_{q_j}) (c_{q_j\sigma}(0) - iV_{q_j} c_{q_j\sigma}(s)) \right. \\
\left. + \alpha V_{q_j} \Delta_j c_{q_j\uparrow}(s) - i\alpha \Delta_j c_{q_j\downarrow}(0) \right], \quad (14)
\]

where $\alpha = (+(-)$ for $\sigma = \uparrow (\downarrow)$. Using these formulas for $c_1(s)$ and $c_{q_j\sigma}(s)$ we can analytically determine the QD occupancy, pairing parameter, subgap currents and its differential conductance.
In the following we set $\hbar = k_B = 1$ and make use of the wide-band limit approximation. All numerical calculations shall be performed for $\Gamma_S = \Gamma_{S_1} = \Gamma_{S_2}$ and $\mu_N = 0$, unless stated otherwise. The energies, currents, and time are expressed in units of $\Gamma_S$, $e\Gamma_S/\hbar$, and $h/\Gamma_S$, respectively.

Assume we have the chemical potentials of superconducting leads $\mu_{S_1} = \mu_{S_2} = 0$ to be grounded. For experimentally available values of $\Gamma_S$, $\Gamma_S \sim 200 \text{ meV}$ [82–84], the typical time and current units would be $\sim 3.3 \text{ ps} \text{ and } 48 \text{ nA}$, respectively.

III. QUANTUM DOT OCCUPANCY

Let us consider the time-dependent QD occupancy after abrupt coupling (at $t = 0^+$) to the normal and superconducting electrodes. We assume no bias voltage between electrodes and make use of the wide band limit approximation and impose $|\Delta_j| = \infty$. Under these assumptions the QD occupation, $n_{\sigma}(t)$, reads (cf. [80] for N-QD-S and [98] for N-QD-N systems):

$$n_{\sigma}(t) = \mathcal{L}^{-1} \left\{ s + i e_{\sigma} + \frac{\Gamma_N}{2} (s - s_2)(s + i e_{\sigma}) \right\} (t) \mathcal{L}^{-1} \left\{ s - e_{\sigma} + \frac{\Gamma_N}{2} (s - s_2)(s - s_4) \right\} (t) n_{\sigma}(0)$$

$$+ \frac{\Gamma_2}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_1)(s - s_2)} \right\} (t) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)} \right\} (t) (1 - n_{-\sigma}(0))$$

$$+ \sum_{k_1,k_2} V_{k_1} V_{k_2} \mathcal{L}^{-1} \left\{ \frac{s + i e_{\sigma} + \frac{\Gamma_N}{2}}{(s - s_1)(s - s_2)(s - i e_{k_1})} \right\} (t) \mathcal{L}^{-1} \left\{ \frac{s - i e_{\sigma} + \frac{\Gamma_N}{2}}{(s - s_3)(s - s_4)(s + i e_{k_2})} \right\} (t) (c_{k_1\sigma}^+(0)c_{k_2\sigma}(0))$$

$$+ \frac{\Gamma_2}{4} \sum_{k_1,k_2} V_{k_1} V_{k_2} \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_1)(s - s_2)(s + i e_{k_1})} \right\} (t) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)(s - i e_{k_2})} \right\} (t) (c_{k_1\sigma}(0)c_{k_2\sigma}^+(0)), \tag{15}$$

where $s_{1,2} = \frac{1}{2} (i e_{\sigma} - e_{\sigma}) - \Gamma_N \pm i\sqrt{5}$, and for $\sigma = \downarrow$ one should replace $(s_1, s_2) \leftrightarrow (s_3, s_4)$, respectively. The first two terms describe the transient QD charge oscillations which depend on the initial QD occupations. The last two terms (with the sums over $k$) are related to the normal lead and they give nonvanishing and nonoscillating contribution to $n_{\sigma}(t)$, regardless of the initial conditions. Note that in Eq. (15) the terms involving the expectation values of the product of electron annihilation and creation operators $c_{q\sigma}$ and $c_{q\sigma}^+$ of the superconducting lead electrons do not appear. Such terms take, e.g., the following integral form (cf. Ref. [80]):

$$\frac{\Gamma_S}{2\pi} \int_{-\infty}^{+\infty} d\epsilon f_{\epsilon}(\epsilon) \mathcal{L}^{-1} \left\{ \frac{s + i e_{\sigma} + \frac{\Gamma_N}{2}}{(s - s_1)(s - s_2)(s - i e_{\sigma})} \right\} (t) \mathcal{L}^{-1} \left\{ \frac{s - i e_{\sigma} + \frac{\Gamma_N}{2}}{(s - s_3)(s - s_4)(s + i e_{\sigma})} \right\} (t), \tag{16}$$

where $f_{\epsilon}(\epsilon)$ is the Fermi distribution function. It is easy to check numerically that the above integral over energy is smaller and smaller with increasing $|\Delta_j|$. Thus in our calculations for $|\Delta_j| = \infty$ we can neglect all terms involving operators $\hat{c}_{q\sigma}(0)$.

The formula (15) can be further elaborated and after some algebra one rewrites the two first terms explicitly while the third and fourth terms can be expressed by integrals over the energy in the normal lead spectrum

$$n_{\sigma}(t) = e^{-\Gamma_S t} \left[ n_{\sigma}(0) + (1 - n_{\sigma}(0) - n_{-\sigma}(0)) \sin^2 \left( \frac{\sqrt{5}t}{2} \right) \frac{\Gamma_{12}}{\delta} \right]$$

$$+ \frac{\Gamma_2}{2\pi} \int_{-\infty}^{+\infty} d\epsilon f_{\epsilon}(\epsilon) \mathcal{L}^{-1} \left\{ \frac{s + i e_{\sigma} + \frac{\Gamma_N}{2}}{(s - s_1)(s - s_2)(s - i e_{\sigma})} \right\} (t) \mathcal{L}^{-1} \left\{ \frac{s - i e_{\sigma} + \frac{\Gamma_N}{2}}{(s - s_3)(s - s_4)(s + i e_{\sigma})} \right\} (t)$$

$$+ \frac{\Gamma_2}{8\pi} \Gamma_{12} \int_{-\infty}^{+\infty} d\epsilon (1 - f_{\epsilon}(\epsilon)) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_1)(s - s_2)(s + i e_{\sigma})} \right\} (t) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)(s - i e_{\sigma})} \right\} (t). \tag{17}$$

Here $f_{\epsilon}(\epsilon)$ is the electron Fermi distribution function for the normal lead and for $\sigma = \downarrow$ the replacement $(s_1, s_2) \leftrightarrow (s_3, s_4)$ should be done. The phase difference $\phi = \phi_1 - \phi_2$ enters Eq. (17) only through the function $\cos \phi$, therefore the QD occupancy satisfies the symmetry relation $n_{\sigma}(\phi) = n_{\sigma}(\phi + 2\pi)$. Note that the part which depends on the initial QD filling oscillates with the period $2\pi / \sqrt{5}$. These oscillations depend on the QD electron energies, $e_{\sigma}, e_{\downarrow}$, both couplings $\Gamma_{S_1}, \Gamma_{S_2}$, and the phase difference $\phi$ of the superconducting order parameters, $\phi = \phi_1 - \phi_2$. The oscillations are damped due to the exponential factor $e^{-\Gamma_S t}$ and in the asymptotic time limit the information about the initial QD occupation is entirely washed out. From Eq. (17) we infer that, when QD is coupled only to the superconducting leads and the initial conditions are $n_{\sigma}(0) = (1, 0)$ or $(0, 1)$, the time-dependent QD occupancy does not change at all (independently of $\phi$ and $\Gamma_{S_1/2}$).

In this case the QD is occupied only by one electron which cannot be exchanged with the superconducting reservoirs due to the infinity large energy gaps. For the initial conditions $n_{\sigma}(0) = (1, 1)$ or $(0, 0)$ the QD occupancy oscillates with the time period $T = \frac{2\pi}{\sqrt{5}}$ for $\phi \neq \pi$ independently of $\Gamma_{S_1/2}$ or for $\phi = \pi$, $\Gamma_{S_1} \neq \Gamma_{S_2}$. These oscillations, however, disappear for $\phi = \pi$ and $\Gamma_{S_1} = \Gamma_{S_2}$, as shown in Fig. 2.

The formula (17) for $\Gamma_N = 0$ resembles the Rabbi oscillations of a typical two-level quantum system described by the effective Hamiltonian $H_{\text{eff}} = \frac{\Gamma_N}{2} (\Gamma_S e^{i\phi} + \Gamma_S e^{i\phi}) \sigma^z + \text{H.c.} + \sum_{\sigma} \epsilon_{\sigma} n_{\sigma}$. Assuming that at $t = 0$ the QD is empty, $n_{\sigma}(0) = 0$, we can calculate the probability $P(t)$ of finding
the QD in the doubly occupied configuration, \( n_1 = n_\downarrow = 1 \).

Within the standard treatment of a two-level system we have [80,99]

\[
P(t) = \frac{\Gamma_{12}}{\Gamma_{12} + (E_1 - E_2)^2} \sin^2 \left( \sqrt{\frac{\Gamma_{12}}{\Gamma_{12} + (E_1 - E_2)^2}} t \right),
\]

where \( E_1 = 0 \) and \( E_2 = \varepsilon_\uparrow + \varepsilon_\downarrow \) are energies of the empty and double occupied configurations, respectively. This formula can be rewritten as \( P(t) = \frac{\Gamma_{12}}{\Gamma_{12} + (E_1 - E_2)^2} \sin^2 \left( \sqrt{\frac{\Gamma_{12}}{\Gamma_{12} + (E_1 - E_2)^2}} t \right) \) and becomes identical with our expression (17) obtained for \( n_\sigma(0) = 0, \Gamma_N = 0 \).

To illustrate such analytical results and to reveal influence of the phase difference of two superconducting leads on the QD occupation in Fig. 2 we show \( n_\uparrow(t) \) and \( n_\downarrow(t) \) with respect to time and \( \phi \) for \( \varepsilon_\sigma = 0 \) (upper panel) and for the Zeeman splitting \( \varepsilon_\sigma = -\varepsilon_\sigma \approx 0.5 \) (bottom panel). We consider here the symmetric coupling \( \Gamma_{\uparrow \downarrow} = \Gamma_{\downarrow \uparrow} = \Gamma_S \) and assume the initial conditions \( n_\sigma(0) = 0, 0 \). Note that for \( \varepsilon_\sigma = 0 \) the QD occupancy becomes spin independent, i.e., \( n_\sigma(t) = n_{\uparrow\downarrow}(t) \) [see Eq. (17)]. For \( t \to \infty \) it always tends to 0.5, regardless of the superconducting phase difference. In the absence of any phase difference we observe the oscillations of \( n_\sigma(t) \) with the period \( T = \pi / \Gamma_S \) which are damped according to the exponential function \( e^{-\Gamma_S t} \). Notice that the period of these oscillations is twice as short compared to the oscillations in the N-QD-S system [80]. For \( \phi \neq 0 \) these oscillations are characterized by the phase-dependent period \( T = \pi / |\Gamma_S| \cos(\phi/2)| \). For the special case \( \phi = \pi \) (\( \Gamma_{\uparrow \downarrow} = 0 \)) the oscillations disappear and the QD charge develops in time exactly in the same way as for the QD coupled only to the normal lead (with \( \varepsilon_\sigma = 0 \)), e.g., Ref. [1]:

\[
n_\sigma(t) = n_\sigma(0) e^{-\Gamma_N t} + \frac{\Gamma_N}{\pi} e^{-\Gamma_N t/2} \int_{-\infty}^{+\infty} d\epsilon f_N(\epsilon) [\cosh(\Gamma_N t/2) - \cos(\epsilon t)]/\Gamma_N^2 t^2.
\]

For \( \mu = 0 \) and the zero temperature case we obtain \( n_\sigma(t) = \frac{1}{2} + e^{-\Gamma_N t} (n_\sigma(0) - \frac{1}{2}) \). It means that for \( n_\sigma(0) = (0, 0) \) or \((1,1)\) the QD occupation increases or decreases monotonically in time without any oscillations, changing from zero (one) to 0.5 (see Fig. 2, upper panel).

The situation changes in the presence of the Zeeman splitting (bottom panel). For symmetric splitting around \( \mu_N = 0 \), \( \varepsilon_\uparrow = -\varepsilon_\downarrow \), the first term of Eq. (17) depends only on the phase difference \( \phi \) and \( \Gamma_S \). Its contribution to the final QD occupancy is the same for arbitrary values of \( \varepsilon_\sigma \). On the other hand the two last terms in Eq. (17) depend separately on \( \varepsilon_\sigma \). For \( \phi = \pi \) the contribution from these terms is identical with the case of the QD coupled only to the normal lead. For \( t = 40 \) and \( \varepsilon_\uparrow = -\varepsilon_\downarrow = 0.5 \) (bottom panel in Fig. 2), the contribution for spin up (down) is \( \sim -0.03 \) (\( \sim 0.95 \)). For \( \phi = 0 \) such contributions become \( \sim 0.49 \) and \( \sim 0.51 \), respectively. One can thus control the QD occupancy by changing the phase difference \( \phi \).

Let us analyze more carefully variation of the QD occupancy against the phase difference \( \phi \). In Fig. 3 (upper panel) we present the ABS energies of the proximitized QD, \( E_{\alpha\beta} = E_\alpha - \epsilon_\beta, (\alpha = \pm, \beta = \pm \equiv \uparrow, \downarrow) \), where \( E_\alpha = \frac{1}{2}(\epsilon_\uparrow + \epsilon_\downarrow) + \alpha \sqrt{\frac{(\epsilon_\uparrow - \epsilon_\downarrow)^2}{4} + \frac{\Gamma_\Sigma^2}{4}} \cos^2 \frac{\phi}{2} \) is the quasiparticle energy, representing a superposition of the empty and double occupied states [100]. In the lower panel we show the QD occupancies \( n_\uparrow(t), n_\downarrow(t) \) and the difference \( n_\uparrow(t) - n_\downarrow(t) \) for \( \Gamma_N = 0.02 \) obtained for particular times \( t \). QD occupancy rapidly changes for such values of \( \phi \) which satisfy the relation \( E_{\uparrow\uparrow} = E_{\downarrow\downarrow} \), i.e., for \( \phi = \pi \pm \arccos \frac{\Gamma_N}{\Gamma_S} \) (here \( \varepsilon_\uparrow + \varepsilon_\downarrow = 0, \varepsilon_\uparrow > 0 \)). Exactly for such values of \( \phi \) we observe an abrupt change of the QD magnetization, which is well visible especially in the long-time (steady) limit. In our case for \( \Gamma_N = 0.02 \) this time equals 200 u.t. (approximately equal to 2.5\( \pi \)). For stronger couplings \( \Gamma_N \) such changeover of the QD magnetization is also observed (not shown here) although it is more smeared around \( \phi = \pi \pm \pi/3 \) even for longer time after the quench.

At a very early stage of the time evolution such a transition of the magnetization from zero value to 1 is only weakly manifested (see the upper thick red curve in the lower panel of Fig. 3). On the other hand, oscillations of the QD occupancies hardly detect the existence of this transition. However, already for \( t \approx \frac{\pi}{\Gamma_S} \approx 50 \) u.t. this transition is well marked on the QD occupancy curves as well as on \( n_\uparrow(t) - n_\downarrow(t) \). Notice the decreasing amplitude and increasing frequency of the QD occupations versus time. These transient characteristics are described by the factor \( \sin^2(2\Gamma_S t) \cos(\phi/2) |e^{-\Gamma_N t}| \), see the first term of Eq. (17). Let us emphasize, that despite oscillatory character of \( n_\sigma(t) \), the resulting magnetization \( n_\uparrow(t) - n_\downarrow(t) \) is a smooth function of \( \phi \).

IV. INDUCED ON-DOT PAIRING

We shall now calculate the pairing amplitude \( \chi(t) \equiv \langle c_\uparrow(t)c_\downarrow(t) \rangle \) driven by the proximity effect, assuming absence
of any bias voltage ($\mu_N = 0$). Using the expressions for $c_1(s)$ and $c_\downarrow(s)$ obtained in Sec. II we find

$$\chi(t) = \frac{i}{2} \left[ \Gamma_S e^{i\phi} + \Gamma_S e^{i\phi} \right] \left[ -n_0(0) \left\{ \frac{1}{(s-s_1)(s-s_2)} \right\} + \frac{1}{(s-s_3)(s-s_4)} \right] \left( t \right) \times \frac{1}{(s-s_3)(s-s_4)} \left( t \right) + \frac{\Gamma_N}{2\pi} \Phi_\uparrow \right]$$

where

$$\Phi_\sigma = \int_{-\infty}^{+\infty} d\varepsilon \mathcal{L}^{-1} \left\{ \frac{1}{(s-s_1)(s-s_2)(s+i\varepsilon)} \right\} \left( t \right) \times \mathcal{L}^{-1} \left\{ \frac{s+i\varepsilon_\sigma + \Gamma_N/2}{(s-s_3)(s-s_4)(s-i\varepsilon)} \right\} \left( 1 - f_N(\varepsilon) \right) \left( t \right) - \int_{-\infty}^{+\infty} d\varepsilon f_N(\varepsilon) \mathcal{L}^{-1} \left\{ \frac{s+i\varepsilon_\sigma + \Gamma_N/2}{(s-s_3)(s-s_4)(s-i\varepsilon)} \right\} \left( t \right) \times \mathcal{L}^{-1} \left\{ \frac{1}{(s-s_3)(s-s_4)(s+i\varepsilon)} \right\} \left( t \right).$$

![Fig. 3. Energies of the subgap quasiparticles of the proximitized QD, $E_{\uparrow\downarrow}^\delta$, (upper panel) and QD occupancies: $n_\uparrow$, $n_\downarrow$, and $n_\uparrow - n_\downarrow$ (solid black, broken red, and thick red curves, respectively) as a function of $\phi$. The occupancies are obtained for $t = 10$, $t = 50$ (shifted down by 1.5) and for $t = 200$ u.t. (shifted down by 3.0). The vertical black lines indicate characteristic points for $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$, and the other parameters are $\epsilon_1 = -\epsilon_\downarrow = 0.5$, $\Gamma_{S1} = \Gamma_{S2} = \Gamma_S = 1$, $\Gamma_N = 0.02$, $\mu_N = 0$.](image3.png)

![Fig. 4. The real (upper panel) and imaginary (bottom panel) parts of the QD induced pairing $\chi(t) = \langle c_\uparrow(t) c_\downarrow(t) \rangle$ as a function of time and the phase difference $\phi_1 - \phi_2$. The bold green line for $t = 40$ in the upper panel corresponds to the case $\epsilon_1 = -\epsilon_\downarrow = 0.5$. The terms proportional to $\epsilon_\uparrow$ and $\epsilon_\downarrow$ should be made for $\sigma = \downarrow$. The terms proportional to $n_\uparrow(0)$ and $(1 - n_\downarrow(0))$ in the above relation can be expressed analytically, and](image4.png)
vanishes when the QD is filled by a single electron at the initial time \( t = 0 \). On the other hand, the real part of \( \chi \) is a nonvanishing function irrespective of the initial conditions. It is worth mentioning that for \( \Gamma_N = 0 \) (i.e., Josephson junction setup) and for \( \varepsilon_1 + \varepsilon_\downarrow = 0 \) the expression for \( \chi(t) \) becomes purely imaginary and is characterized by undamped oscillations inducing d.c. current (see Sec. V). In general, from the analysis of Eq. (20) we infer that the QD induced pairing satisfies the symmetry relation \( \chi(t, \phi) = \chi(t, \phi + 4\pi) \). In particular, for \( t = \infty \), it becomes symmetric with respect to \( \phi = 2\pi \).

V. SUBGAP CURRENTS

Let us consider the currents \( j_{N\sigma}(t) \) and \( j_{S\sigma}(t) \) flowing between the QD and the normal or superconducting electrodes, respectively. These currents depend on time due to the abrupt coupling of all parts of the considered system. For \( t > 0 \), even at zero bias voltage, there are induced transient currents. Such electron currents can be obtained from the evolution of the total number of electrons of the corresponding electrode [1]. For the normal lead we can express it as [38, 50, 51, 57, 68]:

\[
j_{N\sigma}(t) = 2\pi \left( \sum_k V_k \sigma \omega e^{-i\omega t} \epsilon_\sigma(t) \right) - \Gamma_N n_{\sigma}(t), \quad (24)
\]

where we have assumed the energy-independent normal lead spectrum. Using the formulas of Sec. II we find

\[
j_{N\sigma}(t) = \frac{\Gamma_N}{\pi} \int_{-\infty}^{+\infty} d\varepsilon f_{N\sigma}(\varepsilon) e^{-i\varepsilon t} \times \mathcal{L}^{-1} \left\{ \frac{s + i\varepsilon_{-\sigma} + \Gamma_N/2}{(s - s_1)(s - s_2)(s - i\varepsilon)} \right\}(t) - \Gamma_N n_{\sigma}(t).
\]

(25)

Inserting the inverse Laplace transform and using the expression for \( n_{\sigma}(t) \) one can obtain the analytical relation for \( j_{N\sigma}(t) \). However, this solution for arbitrary \( t \) cannot be written in relatively compact (or transparent) form, so we restrict ourselves to the asymptotics \( t \to \infty \)

\[
j_{N\sigma} = \frac{\Gamma_N}{\pi} \int_{-\infty}^{+\infty} d\varepsilon \left\{ f_{N\sigma}(\varepsilon) \left[ \Gamma_N \left( \frac{i\varepsilon + \varepsilon_{-\sigma}}{\Gamma_N} \right) \left( \frac{\varepsilon_{+\sigma}}{\Gamma_N} \right) \right] - \frac{\Gamma_N}{2} \left( \frac{\varepsilon_{+\sigma}}{\Gamma_N} \right) (\varepsilon_{+\sigma}^2 + \varepsilon_{-\sigma}^2) \right\}
\]

(26)

\[
\times \left\{ 1 - f_{N\sigma}(\varepsilon) \right\} \frac{\Gamma_N\Gamma_{12}}{8 \left( \frac{\varepsilon_{+\sigma}^2}{\Gamma_N} + \varepsilon_{-\sigma}^2 \right) \left( \varepsilon_{+\sigma}^2 + \varepsilon_{-\sigma}^2 \right)} \right\},
\]

where \( \epsilon_{\sigma\beta} = \varepsilon + E_{\sigma\beta} \) and \( E_{\sigma\beta} \) are the quasiparticle energies of the proximitized QD.

Upper panel of Fig. 5 shows the time-dependent current flowing from the normal lead to the QD as a function of the phase difference \( \phi = \phi_1 - \phi_2 \) obtained for the unbiased system and \( \varepsilon_{\sigma} = 0 \). At the beginning the current starts to flow from the normal electrode to the empty QD. In a next stage, electrons tunnel in both directions with the characteristic oscillations. These damped oscillations are clearly visible and for \( t \to \infty \) the current vanishes for all \( \phi \). The period of these oscillations increases with \( \phi \), similarly to the behavior observed for the QD occupancy. Exceptionally, for \( \phi = \pi \), the current tends to its asymptotic value without any oscillations according to the formula (valid for the zero temperature, \( \varepsilon_{\sigma} = 0 \) and \( \Gamma_1 = \Gamma_2 \))

\[
j_{N\sigma}(t) = \Gamma_N \epsilon_{\sigma}(\varepsilon_{-\sigma}^2 - n_{\sigma}(0)). \quad (27)
\]

We can notice that right after the abrupt coupling (at \( t = 0^+ \)) the large value of transient current \( j_{N\sigma} \) is induced in the system (\( \sim \frac{\Gamma_N}{\pi} \)) which is artifact of the WBL approximation [15]. We have checked that by applying a more realistic (smooth) QD-leads coupling profile the initial current would gradually increase, revealing the same period of oscillations and other overall features [80].

The situation looks a bit different for the currents flowing between the QD and superconducting leads. To calculate these currents we start from the standard formula \( j_{S\sigma}\sigma(t) = 2\pi \left( \sum_{q \neq \eta} V_{q\eta} \epsilon_{q\eta}(\varepsilon_{\sigma} + \varepsilon_{\eta})(t) \right) \) and use the Laplace transforms for \( c_{\sigma}^+ \) and \( c_{q\eta}(s) \) obtaining [80]

\[
j_{S\sigma}(t) = \Gamma_{N}\epsilon_{\sigma} e^{-\Gamma_{N}/(s - n_{\sigma}(0)) \mathcal{L}^{-1}},
\]

(28)
As usually, the replacement \((s_1, s_2) \leftrightarrow (s_3, s_4)\) should be made for \(\sigma = \downarrow\). After straightforward algebra we can derive more explicit form for the superconducting current

\[
J_{S_1,\sigma}(t) = \frac{\Gamma_{S_1}S}{2}\sqrt{3}\left(1 - n_\sigma(0) + n_{-\sigma}(0)\right)e^{-\Gamma_{xt}}\left[\frac{1}{\Gamma_{S_1}}\sin(\sqrt{3}t) \mp \Gamma_{S_1}(\epsilon_\sigma + \epsilon_{-\sigma})\sin\phi(1 - \cos(\sqrt{3}t))\right]
\]

\[
+ \frac{\Gamma_{N}S\pi}{\sqrt{3}}\left\{ \left(\Gamma_{S_1} + \Gamma_{S_2}\right)e^{\pm i\phi}\Phi_{\sigma}\right\}.
\]

(29)

Using the relation for the induced pairing, Eq. (20), the above current can be recast as \(J_{S,\sigma}(t) = \left(\Gamma_{S_1}e^{\pm i\phi}(c_\sigma(1)c_\chi(1))\right)^*\). Assuming that \(\langle c_\sigma(1)c_\chi(1)\rangle = \langle |c_\chi(1)c_\chi(1)|\right)e^{\pm i\phi}\), where \(\phi_d\) is the argument (phase) of \(\langle c_\chi(1)c_\chi(1)\rangle\), we obtain (e.g., Ref. [69]):

\[
J_{S,\sigma}(t) = \Gamma_{S_1} \left| \langle c_\sigma(1)c_\chi(1)\rangle \right| \sin(\phi_d - \phi_d),
\]

where \(j = 1, 2\). Inspecting (30) we conclude that the currents flowing between the QD and a given superconducting lead do not depend on spin, \(J_{S,\sigma}(t) = J_{S_{-\sigma},\sigma}(t)\), irrespective of the spin dependent QD energy levels. This is a consequence of the fact that the QD can exchange charge with the superconducting leads only via pairs of opposite spin electrons.

Formula (30) simplifies for the case \(\Gamma_{S_1,\sigma} = \Gamma_{S_{-\sigma}} \equiv \Gamma_S\) and \(\epsilon_1 = \epsilon_{-1} = 0\), when we obtain

\[
J_{S,\sigma}(t) = \Gamma_{S_1}e^{\Gamma_{xt}}[1 - n_\sigma(0) - n_{-\sigma}(0)]
\]

\[
\times \cos(\phi/2)\sin(2\Gamma_S/\cos(\phi/2)/t)
\]

\[
+ \frac{\Gamma_{N}S\pi}{\sqrt{3}}\cos(\phi/2)\sin\phi\left[\Phi_{\sigma}\right] - \frac{\Gamma_{N}S\pi}{\sqrt{3}}\sin\phi\left[\Phi_{\sigma}\right].
\]

(31)

For \(\mu_N = 0\) the real part of \(\Phi_{\sigma}\), \(\Re\left\{\Phi_{\sigma}\right\}\), vanishes [80] and in such a case for \(\phi = \pi\) and identical couplings to both superconducting leads the currents \(J_{S,\sigma}(t)\) vanish.

Under nonequilibrium conditions (\(\mu_N \neq 0\)) for symmetric couplings and assuming \(\epsilon_1 = -\epsilon_{-1}\), the asymptotic \((t \to \infty)\) value of the superconducting current can be expressed as

\[
J_{S,\sigma} = \frac{\Gamma_{S}^{2}\Gamma_{S}}{4\pi} \int \frac{(1 - f_{N}(\epsilon))d\epsilon}{\left[\left(\frac{\Gamma_{S}}{2\Gamma_{S}}\right)^{2} + \epsilon^{2} - \epsilon_{-1}^{2}\right]} - \frac{\Gamma_{N}S\pi}{\sqrt{3}}\int \frac{f_{N}(\epsilon)\sin(\phi/2)d\epsilon}{\left[\left(\frac{\Gamma_{S}}{2\Gamma_{S}}\right)^{2} + \epsilon^{2} - \epsilon_{-1}^{2}\right]} - \frac{\Gamma_{N}S\pi}{\sqrt{3}}\int \frac{f_{N}(\epsilon)\sin(\phi/2)d\epsilon}{\left[\left(\frac{\Gamma_{S}}{2\Gamma_{S}}\right)^{2} + \epsilon^{2} - \epsilon_{+1}^{2}\right]}
\]

where \(\epsilon_{\alpha\beta} = \epsilon + E_{\alpha\beta}\) and \(E_{\alpha\beta}\) denote quasiparticle energies of the proximitized QD. Notice that the term in the above formula vanishes for zero temperature and \(\mu_N = 0\). In this case the superconducting current can be rewritten to the following (Josephson-like) formula

\[
J_{S,\sigma} = \frac{\Gamma_{S}}{4\pi} \sin\phi\left[\arctan\left(\frac{\epsilon_1^{2} + \frac{\Gamma_{S}}{2\Gamma_{S}} + \epsilon_{-1}^{2} - \epsilon_{+1}^{2}}{(\Gamma_{S}\Gamma_{N})\cos(\phi/2)}\right) - \frac{\pi}{2}\right].
\]

(33)

Let us remark that the formula for the current, Eq. (31), can be used to determine the coupling value \(\Gamma_S\). As the time oscillations are described by the first term of Eq. (31), then for a given \(\phi\) the oscillating part of \(J_{S,\sigma}(t)\) is proportional to \(\sin(2\Gamma_S/\cos(\phi/2)t)\). The period of these oscillations \(T = \frac{\pi}{\Gamma_S(\cos(\phi/2)/t)}\) for the system characterized by a sufficiently small \(\Gamma_S\) and \(\phi \approx \pi\) should be experimentally detectable.

Lower panel in Fig. 5 presents the current \(J_{S_1}(t)\) as a function of \(n_0(\epsilon) = n_{-1}(\epsilon) = 0\). The current oscillates with a damping amplitude and for large time it tends to a nonzero asymptotic value given in Eq. (33). The asymptotic value of the current does not depend on the initial QD occupancies, see Eq. (29). However, the transient currents are different for the QD initial occupancies, \(n_0(\epsilon) = (0, 0), (1, 1)\) and for \(n_0(\epsilon) = (0, 1), (1, 0)\). In the first case the current indicates a rather rich time-dependent structure before it attains the asymptotic value. This is a consequence of the Rabi-like oscillations (damped via \(e^{-\Gamma_{xt}}\) due to the coupling with normal lead) between the empty and double occupied QD configurations and is described by the first term of Eq. (29) which depends on the factor \((1 - n_\sigma(0) - n_{-\sigma}(0))\). This factor disappears for the initial occupancies \(n_0(\epsilon) = (0, 1)\) and \((1, 0)\) and all time dependence of \(J_{S,\sigma}(t)\) is described by the last term of Eq. (29).

This term, however, in contrast to the former case does not introduce any visible oscillations for small \(\Gamma_N\) but enters the formulas for \(J_{S,\sigma}\), irrespective of the initial conditions. From Fig. 5 we can learn that at a short time after the quench the current is symmetric with respect to \(\phi = \pi\). This symmetry, however, is quickly lost in the long time scale.

In Fig. 6 we present time dependent currents \(J_{S_1,\sigma}\) and \(J_{S_1,\sigma}\) vs the phase difference \(\phi\) for the finite Zeeman splitting of energy levels, \(\epsilon_1 = -\epsilon_{-1} = 0.5\Gamma_S\). Both currents oscillate with the period dependent on the phase difference \(\phi\). As before, this period increases with \(\phi\) and for \(\phi = \pi\) the currents do not flow in the system. Comparing such \(\phi\) dependence of the currents with those presented in Fig. 5 (lower panel) for \(\epsilon_1 = 0\) we observe a different behavior, especially at asymptotic large time. In the presence of the Zeeman splitting the asymptotic currents almost vanish for some \(\phi\) interval around \(\phi = \pi\). To study this effect in more detail we show in Fig. 6, bottom panel, the superconducting currents for several values of the Zeeman splittings (solid lines for \(\epsilon_1 = -\epsilon_{-1} = 0, 0.25, 0.5, 0.75, \) and 1.0 expressed in units of \(\Gamma_S\)). As one can see, in the absence of the Zeeman splitting the current does not flow only for \(\phi = 0, \pi\). In the presence of the Zeeman term the zero-value superconducting current interval of \(\phi\) increases, but at the same time the maximal values of the currents diminish. For \(\epsilon_1 = -\epsilon_{-1} \gg 1\) the superconducting currents do not flow. The corresponding asymptotic occupancies of the QD, \(n_1(\phi, t = \infty)\), are shown in Fig. 6, bottom panel (broken lines). One can notice that
the occupancies decrease monotonically with φ and remain very low for the zero-current interval of φ. The changes of the QD occupancies in the presence of the Zeeman splitting reflect phasal dependence of the superconducting currents. These changes are coordinated with the QD magnetization and will be discussed in the next paragraph (compare also φ dependence of n↑ for ε1 = −ε↓ = 0.5 and ΓN = 0.1 shown in the lower panel of Fig. 3).

In Fig. 7 we analyze the time dependence of jS,φ(t) for some selected values of time t, starting from the quench at t = 0 until nearly the asymptotically large times. In the lower panel, ΓN = 0.02, the φ dependence of the current demonstrates abrupt changing of the current value at points corresponding to E± = E− (see upper panel in Fig. 3). These jumps of the current are clearly visible for large times. However, for larger ΓN, e.g., for ΓN = 0.1 (upper panel, Fig. 7) the φ dependence of the current even for asymptotically large times does not show such sharp changes. Notice that the time at which the current achieves constant (in time) values is much shorter in comparison to the case of ΓN = 0.02. In both regimes of ΓN we can estimate this time as 1/ΓN [compare the results for φ dependence of n↑(t)]. For small time the abrupt change of the current is not visible but for larger time it becomes evident in spite of the oscillations. Such a transition is very well visible in the asymptotics, where the oscillations vanish. For larger ΓN the current tends to its asymptotic value (without time oscillations) in much shorter time than for smaller ΓN, due to the damping factor e−ΓN t [see the first term in Eq. (31)].

Let us consider the simple case of the QD coupled solely to superconducting leads, assuming ΓS, = ΓS, = ΓN, ε↑ = −ε↓ and n↑(0) = n↓(0) = 0. In this case

\[ n_\sigma(t) = \sin^2(\Gamma_S \cos(\phi/2)t), \]

(34)

\[ j_{S,\phi}(t) = \frac{\Gamma_S}{2} \cos(\phi/2) \sin(2\Gamma_S |\cos(\phi/2)|t). \]

(35)

The QD occupancy and the current do not depend on spin and, in addition, both superconducting currents, jS,φ, are exactly identical. Note, however, that for ε↑ + ε↓ ≠ 0 these currents differ one from another, see Eq. (29), and their difference equals ΓS \sin \phi (1 - \cos(\sqrt{3} t))(\epsilon_\phi + \epsilon_\delta)/\delta. The current jS,φ vanishes for φ = π and ΓS = ΓS,. For different couplings, ΓS, ≠ ΓS,, the current does not vanish, even for φ = π. For instance jS,φ in this case (for εα = 0) is found to be jS,φ(t) = \frac{\Gamma_S}{2} \sin[(\Gamma_S - \Gamma_S,) t].

It would be interesting to consider the transition from the permanently oscillating superconducting currents in the system of the QD placed only between two superconducting leads (ΓN = 0) to finite constant asymptotic values of such currents in the presence of the third normal lead (ΓN ≠ 0), see, e.g., the bottom panel in Fig. 6. From Eq. (31) we see that for ΓN ≠ 0 the current consists of two parts. The first one corresponds to the transient oscillations damped by the factor e−ΓN t, whereas the second one is described by the imaginary part of Φσ. This part of the current slowly evolves
of and less effectively, and simultaneously the imaginary part \( \Gamma_1 \)
Asymptotic value of this current vanishes with decreasing
system parameters are the same as in Fig. 2 with
as a function of the bias voltage
Eqs. (17) and (25), we obtain at zero temperature
lead. Using the expressions for the current and QD charge,
expressing it in units of \( \frac{4e^2}{h} \), Eq. (31).

VI. DIFFERENTIAL SUBGAP CONDUCTANCE

The next part of our studies is devoted to the subgap
time-dependent Andreev conductance \( G_\sigma(\mu, t) \),
expressing it in units of \( \frac{4e^2}{h} \). We investigate this quantity as
function of the bias voltage \( \mu = \mu_N \) applied to the normal
lead. Using the expressions for the current and QD charge,
Eqs. (17) and (25), we obtain at zero temperature

\[
G_\sigma(\mu, t) = \Re \left[ \sum_{\nu} G_N e^{-i\nu t} \mathcal{L}^{-1} \left\{ \frac{s + i\epsilon_{\nu} + \Gamma_N/2}{(s-s_1)(s-s_2)(s-i\mu)} \right\}(t) \right],
\]

\[
+ \frac{\Gamma_1^2}{8} \mathcal{L}^{-1} \left\{ \frac{1}{(s-s_1)(s-s_2)(s+i\mu)} \right\}(t)
\times \mathcal{L}^{-1} \left\{ \frac{s+i\epsilon_{\nu}+\Gamma_N/2}{(s-s_1)(s-s_2)(s-i\mu)} \right\}(t),
\]

where for \( \sigma = \uparrow \) the replacement \( (s_1, s_2) \rightarrow (s_3, s_3) \) should
be made. Notice that for \( \epsilon_{\uparrow} = \epsilon_{\downarrow} \) the conductance is spin
independent \( G_{\uparrow} = G_{\downarrow} = G \).

In Fig. 8 we plot the time-dependent conductance
\( G_\sigma(\mu, t) = G \) as a function of \( \mu \) for different phase difference

\[ t = \frac{\phi}{\tau} \cos(\phi/2), \text{Eq. (31)}. \]

between the superconducting leads, i.e., for \( \phi = 0 \) (upper panel) and for \( \phi = 0.85\pi \) (bottom panel), in the presence of
weakly coupled normal electrode, \( \Gamma_N = 0.1\Gamma_g \) \( (\Gamma_{S1} = \Gamma_{S2} = \Gamma_g = 1) \) and \( \epsilon_{\sigma} = 0 \). The process of forming the Andreev
subgap states is clearly visible. We observe that for \( \phi = 0 \) in
the limit of large time the conductance is characterized by two
well pronounced maxima appearing at \( \mu \approx \pm \Gamma_g \) whose half-widths
gradually shrink in time. These maxima appear after
some time interval after abrupt switching of the QD-leads
connections (we denote such a time scale by \( \tau_1 \), see Fig. 9). This
characteristic time is needed to build up two distinct maxima
of \( G \) and it depends on the phase difference \( \phi \)---compare the
upper and bottom panels in Fig. 8. Time evolution of such
quasiparticle peaks allows us to estimate how fast the Andreev
quasiparticles appear in the system, and thus it is desirable to
study this process in more detail.

By inspecting \( G_\sigma(\mu, t) \) in Fig. 8 we observe that up to
some specific time \( \tau_1 \), a broad one-peaked structure of \( G \) is
present. Then, the conductance rapidly transforms in time into
a two-peak structure. The position of each quasiparticle peak
evolves in time to its steady limit value (that time is called \( \tau_2 \))
and finally the width and height of peaks are established after
the time \( \tau_1 \) (see Fig. 9, bottom panel). In Fig. 9 we display
the position of the quasiparticle peaks maxima vs time and
\( \mu \) for different values of \( \Gamma_N \) and \( \phi \) indicated in the legend
(upper panel). As one can see, the moment of the appearance
of the two-peak structure, \( \tau_1 \), depends on both \( \phi \) and \( \Gamma_N \).
However, for \( \phi = 0 \) this time only slightly depends on \( \Gamma_N \).
With increasing $\phi$ it increases with remarkable dependence on $\Gamma_N$ (for a given $\phi$ it increases with $\Gamma_N$). The time scale for appearance of the two-peak structure is very small and for $\phi = 0$ it equals approximately 2.5 u.t., for $\phi = \pi/2$ it changes from $\sim 3$ u.t. for $\Gamma_N = 0.1$ up to $\sim 4$ u.t. for $\Gamma_N = 0.5$, and for $\phi = 3\pi/4$ it changes from $\sim 6$ u.t. for $\Gamma_N = 0.1$ up to $\sim 8$ u.t. for $\Gamma_N = 0.5$, respectively (see upper panel, Fig. 9). Positions of the maxima versus $\mu$ evolve in time during approximately $\tau_2 \approx 12$ u.t. (the bold parts of lines in the bottom panel) and attain their steady-state values. Note that the asymptotic quasiparticle peaks heights and widths are achieved with the envelope function $1 - \exp(-t/\tau_f)$, where $\tau_f = \Gamma_N/2$ can be deduced from the explicit expression for $G_\sigma(\mu, t)$ in which the long living terms proportional to $\exp(-\Gamma_N t/2)$ are present.

Let us consider a few special cases, for which the simpler analytical formulas can be given. For $\phi = \pi$, $\epsilon_\sigma = 0$, and $\Gamma_S = \Gamma_S$, the conductance takes the form (here $G_1 = G_0 = G$):

\[
G(\mu, t) = \frac{\Gamma_N}{(2 + \mu^2)} \left[ \frac{\Gamma_N}{2} + e^{-\Gamma_N t/2} \cdot \left( \frac{\Gamma_N}{2} \cos(\mu t) - \Gamma_N \cosh \left( \frac{\Gamma_N t}{2} \right) + \mu \sin(\mu t) \right) \right].
\]

(37)

In this case the zero bias conductance reads $G(\mu = 0, t) = 2[1 + e^{-\Gamma_N t/2}(1 - 2 \cosh(\Gamma_N t/2))]$ and for $t = 2 \ln 2/\Gamma_N$ it reaches the optimal value equal to 0.5 and it vanishes for $t \to \infty$.

We notice the vanishing conductance $G_0(\mu, t = \infty)$ for the symmetric couplings $\Gamma_S = \Gamma_S$ obtained for $\phi = \pi$. For $\Gamma_S \neq \Gamma_S$, the conductance is qualitatively different. Assuming $\Gamma_S = \Gamma_S$, we obtain $\Gamma_1 = \Gamma_2 = \Gamma_S(1 + k^2 + 2k \cos \phi)$, so for $k \neq 1$ and $\phi = \pi$ we get $\Gamma_1 = \Gamma_2 = \Gamma_S(1 - k^2)$. In such a case for calculating the conductance we should also take into account a contribution from the second term of Eq. (36). As differential conductance depends on the couplings $\Gamma_1$, $\Gamma_2$, and $\phi$ only through $\Gamma_1$, $\Gamma_2$, therefore different choices of these parameters can lead to the same values of $G_0$. The conductance calculated for $\Gamma_1 = \Gamma_2$ and a given phase difference $\phi$ is identical to the one obtained for arbitrary asymmetric couplings $\Gamma_S = \Gamma_S$ and using the effective phase difference $\phi = \arctan((1 - k^2 + 2 \cos \phi)/2k)$, which satisfies the condition $|1 - k^2 + 2 \cos \phi|/2k| \leq 1$. It means that asymmetry in the couplings to superconducting leads $\Gamma_1$, $\Gamma_2$, can be compensated by the phase difference parameter $\phi$. This conclusion refers also to the QD occupancy and the current flowing between the QD and the normal lead. Since explicit expression for $G_0(\mu, t)$ is rather lengthy, we skip it here and present only its asymptotic form ($t \to \infty$) for $\Gamma_S = \Gamma_S = \Gamma_S$, $\epsilon_1 = -\epsilon_1$ ($G_1 = G_1 = G$)

\[
G(\mu) = \frac{\Gamma_1^2}{2} \cos^2 \left( \frac{1}{2} \phi \right) \left\{ \frac{\Gamma_S^2}{4 + \mu^2} - \frac{1}{2} \frac{\Gamma_S^2}{(4 + \mu^2)} \right\}.
\]

(38)

where $\mu_{ab} = \mu + E_{ab}$. For $\Gamma_N \ll \Gamma_S$ the asymptotic conductance has four maxima placed at $\mu \approx \pm \epsilon_1 \pm \epsilon_2$ or equivalently at $\mu = E_{++}$, $E_{-+}$, $E_{+-}$, and $E_{--}$, respectively. Note that the asymptotic conductivity $G(\mu)$ does not depend on spin but in general $G_0(\mu, t)$ can be spin dependent.

It is also interesting to check influence of the superconducting phase difference on the asymptotic Andreev conductance behavior. For arbitrary $\phi \neq \pi$ and $\epsilon_\sigma = 0$, the asymptotic value of $G_\sigma(\mu, t)$ can be written as follows (for $t \to \infty$)

\[
G(\mu) = \frac{\Gamma_1^2}{2} \Gamma_2^2 \mu \left( 4 \left[ \frac{\Gamma_1^2}{4} + \frac{\Gamma_2^2}{4} \right] + \frac{\Gamma_1^2}{4} \right) \right).}

(39)

Figure 10 presents the asymptotic conductance, $G(\mu, t = \infty)$, as a function of the bias voltage $\mu$, and the phase difference $\phi$. As one can see for $\phi = 0$ two distinct maxima of $G$ are visible (cf. Fig. 8 for $t = 100$). For nonzero $\phi$, which satisfies the condition $\cos(\phi) > \Gamma_1^2 - \Gamma_2^2$, these two maxima appear at points $\mu = \pm \sqrt{\Gamma_1^2 - \Gamma_2^2}$. In the opposite case, there is only one maximum at $\mu = 0$ whose height is reduced to zero value with $\phi \to \pi$. In consequence, for $\phi = \pi$ and $t = \infty$ the conductance vanishes for all $\mu$. Note that for the QD coupled only to one superconducting and one normal electrode, the zero-bias conductance is invariant under the replacement $\Gamma_N \leftrightarrow \Gamma_S$.[70]. However, in our system with two superconducting leads this conclusion is no longer valid, even for the symmetric couplings case, $\Gamma_S = \Gamma_S$. Such a property is achieved only for $\phi = \pi/2$.

In the last part of this section we discuss the time evolution of the ABS for nonzero splitting of the QD energy levels. In the first case we consider the symmetric splitting around the zero energy (Fig. 11, $\epsilon_1 = -\epsilon_1 = 0.5$ for $\phi = 0.85\pi$) and in the second case the splitting is symmetric but around the nonzero energy value equal 0.5 (Fig. 12, $\epsilon_1 = 1$, $\epsilon_2 = 0$ for some specific values of time after the quench). In Fig. 11 we analyze the approach to equilibrium of $G_1(\mu, t)$ for two values of $\Gamma_N$, $\Gamma_N = 0.1(0.02)$ upper (bottom) panel. We show only $G_1(\mu, t)$ as $G_1(\mu, t)$ is symmetric (with respect to $\mu = 0$) in relation to $G_1$. The maxima of $G_1$ for large time correspond to $E_{+-}$, $E_{++}$, $E_{--}$, and $E_{++}$ ABS states (beginning from negative values of the bias voltage $\mu$). It is interesting that the time evolution of $E_{++}$, $E_{++}$ ABS is different from the evolution of $E_{--}$ and $E_{++}$, respectively. The stationary values of the conductance peaks corresponding to $G_1$ and $G_1$ are all
the same [according to Eq. (38)] but the ABS $E_{-+}$ and $E_{++}$ begin to appear later than $E_{--}$ and $E_{+-}$. For $\Gamma_N = 0.1 (0.02)$ this delay time can be approximately estimated as 30 (60) u.t. For stronger coupling $\Gamma_N$ (upper panel) the ABS peaks are wider in comparison to the case of weakly coupled normal electrode (bottom panel) and appear earlier for smaller $\Gamma_N$.

In Fig. 12 we show the phase dependence of $G_t$ and $G_1$, calculated for small time, $t = 10$ u.t. (upper panels), for $t = 30$ u.t. (middle panels), and for long time, $t = 100$ u.t. (bottom panels) at which the conductance attains the stationary values. In addition, in the upper panels the curves representing the localization of the ABS states on the ($\mu, \phi$) plane are depicted. We observe the essential difference with strong asymmetry between $G_t$ and $G_1$ at a short period of time after the quench. The time evolution of $G_t$ ($G_1$) is limited to the appearance of $E_{-+}$ ($E_{++}$) ABS. Next, for larger time other Andreev states appear but the most visible are still the curves corresponding to $E_{-+}$ (for $\sigma = \uparrow$) and $E_{++}$ (for $\sigma = \downarrow$). These states vanish only at relatively large times after the quench.

**VII. CORRELATION EFFECTS**

Finally let us address the correlation effects, driven by the Coulomb interactions $Uc_i^+c_i^+c_i^+c_i^-$ that should be added to the term $H_QD$. Such electrostatic repulsion is usually in conflict with the local electron pairing. In nanoscopic systems, however, their relationship is a bit more subtle. On one hand the Coulomb repulsion $U > 0$ suppresses the magnitude of on-dot pairing potential $\chi(t) = (c_\uparrow(t)c_\uparrow(t))$. In addition to this monotonous behavior, at some critical value $U_c$ there occurs $\pi$ shift of the complex function $\chi(t)$ [101]. Under such circumstances the subgap quasiparticles cross each other and the correlated quantum dot changes configuration of its ground state (singlet-doublet quantum phase transition). When the quantum dot is embedded into Josephson junction geometry, such an effect leads to reversal of $dc$ supercurrent which has been extensively discussed by various groups [94,102].

Salient features of this $\pi$-shift effect can be captured within the lowest order perturbative treatment of the Coulomb term [101]. Crossing of the subgap quasiparticles can be qualitatively obtained using the Hartree-Fock-Bogoliubov decoupling scheme

$$c_\uparrow^+c_\uparrow^+c_\downarrow^+c_\downarrow^+ \simeq n_\uparrow(t)c_\uparrow^+c_\downarrow^+ + n_\downarrow(t)c_\uparrow^+c_\downarrow + n_\uparrow(t)n_\downarrow(t) + \chi(t)c_\uparrow^+c_\downarrow^+ - |\chi(t)|^2. \quad (40)$$

In this mean-field treatment we can absorb the Hartree-Fock terms to the renormalized QD energy level $\epsilon_\sigma + Un_{\sigma}(t)$ and the anomalous (pair source/sink) terms rescale the effective on-dot pairing to $1/2(\Gamma_2\epsilon^{\mu\nu} + \Gamma_2\epsilon^{\nu\mu}) - U\chi(t)$. In a weak coupling regime $U < U_c$ the lowest order approximation (40)
has been shown to qualitatively reproduce the results of more sophisticated many-body techniques [92,101].

Since analytical determination of the time-dependent observables is no longer possible, we have investigated the correlation effects by means of self-consistent numerical calculations. In what follows, we focus on the special case of correlated quantum dot, by means of sophisticated many-body techniques [92,101]. This is a rather obvious fact, considering that Coulomb repulsion has direct influence on the energies of subgap quasiparticles [93]. Furthermore, electron correlations seem to substantially enlarge a temporal region of the quasiparticle buildup (characterized by the time scale $\tau_f$). For a more reliable study of this nontrivial interplay between the transient and correlation effects there should be used some sophisticated (nonperturbative) methods.

**VIII. SUMMARY**

We have investigated the dynamics of subgap quasiparticles for the setup comprising the quantum dot (QD) embedded between two superconducting leads and another metallic electrode. Transient phenomena, caused by an abrupt coupling of the QD to external reservoirs, have been studied solving the Heisenberg equations of motion within the Laplace transform technique which easily incorporates the initial conditions. We have determined the time-dependent charge and magnetization of QD, development of the on-dot pairing, and transient currents induced under the equilibrium (for identical chemical potentials of the leads) and nonequilibrium conditions (i.e., for the biased system). Similar effects can be also observed in the Josephson junctions periodically driven by external fields [103].

In the limit of large energy gap of superconducting reservoirs we have derived analytical formulas for time-dependent observables. We have distinguished between two contributions appearing in expressions for the QD occupancy $n_\sigma(t)$, on-dot pairing amplitude $\langle c_\uparrow(t)c_\downarrow(t)\rangle$, and charge current flowing between the QD and superconducting leads. The first term depends on the initial QD occupancy (but is not dependent on the chemical potentials of external reservoirs) and is responsible for an oscillating transient behavior of the considered quantities, with a characteristic damping $\exp(-\Gamma_N t)$ driven by the QD-normal lead coupling $\Gamma_N$. Contribution of this term is proportional to the factor $(1 - n_\uparrow(0) - n_\downarrow(0))$, so it can vanish for some specific initial QD occupancies $n_\sigma(0)$. The second part appearing in the considered formulas depends mainly on $\Gamma_N$ and it induces a monotonous time dependence of the observables. The latter term is present in all expressions, regardless of the initial conditions.
We have studied dependence of the amplitude and the period of transient oscillations on the model parameters. We have shown that under specific initial conditions (assuming either the empty and doubly-occupied initial QD configurations) these oscillations are reminiscent of Rabi-type transitions, typical for a two-level system, which in our case correspond to a pair of the Andreev states.

We have also checked influence of the phase difference $\phi$ between superconducting reservoirs on transient phenomena. For $\phi = 0$ the asymptotic occupancy of QD seems to be independent on such phase difference. However, in the presence of the Zeeman splitting the occupancies $n_\uparrow(t)$ become sensitive to $\phi$ and reveal an abrupt change of the QD magnetization $n_\uparrow(t) - n_\downarrow(t)$. Right after the quench this feature is obscured by transient phenomena, however, it becomes more and more evident starting from $t \geq \frac{1}{\Gamma_0}$ (Fig. 3). Finally, we have analyzed the time-dependent differential conductance of the charge current induced by the bias voltage applied to the normal lead. Its phase dependence exhibits the two-peak structure (typical for the stationary limit) which gradually develops within a finite time interval. This characteristic time scale increases with respect to the phase difference $\phi$ and can be spin dependent in the presence of the Zeeman splitting. Unexpectedly, we have found a puzzling particle-hole asymmetric development of the subgap quasiparticles, which ultimately become symmetric and spin independent in the asymptotic limit $t \to \infty$.

Since transient currents are measurable by the present-day experimental resolution in the subpicosecond regime, we hope that such spectroscopy could precisely determine all time scales, characteristic for the subgap quasiparticles of the proximitized quantum dots. They could be also extended onto heterostructures with topological superconductors, probing the dynamical properties of more exotic Majorana-type quasiparticles.

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[90] A. Oguri, Y. Tanaka, and J. Bauer, Interplay between Kondo and Andreev-Josephson effects in a quantum dot coupled to


