

# Flow equation approach to the linear response theory of superconductors

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We apply the flow equation method for studying the current-current response function of electron systems with the pairing instability. To illustrate the specific scheme in which the flow equation procedure determines the two-particle Green's functions, we reproduce the standard response kernel of the BCS superconductor. We next generalize this nonperturbative treatment considering the pairing field fluctuations. Our study indicates that the residual diamagnetic behavior detected above the transition temperature in the cuprate superconductors can originate from the noncondensed preformed pairs.

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## I. INTRODUCTION

The nonperturbative scheme of the flow equation method introduced by Wegner<sup>1</sup> and independently by Wilson with Głazek<sup>2</sup> proved to be a useful tool for investigating a number of problems in condensed matter physics,<sup>3</sup> mesoscopic systems,<sup>4</sup> quantum optics,<sup>5</sup> and quantum chromodynamics.<sup>6</sup> This procedure has been also recently applied to study the nonequilibrium transport phenomena of the correlated nanoscopic systems.<sup>7</sup> The main idea is rather simple and relies on a continuous process that, step by step, transforms the Hamiltonian to a diagonal or at least block-diagonal structure. One can use, for this purpose, various operators depending on the subtleties of the discussed problem.<sup>3</sup>

Such a continuous diagonalization scheme is reminiscent of the renormalization group (RG) technique.<sup>8</sup> They are similar with regard to the treatment of high- (low-) energy states [fast (slow) modes]. In initial steps of the continuous transformation procedure, mainly the most off-diagonal terms (i.e., high-energy sector) are dealt with. Subsequently, the remaining parts closer to the diagonal are transformed. Since different energy scales are successively transformed or renormalized one by one, the algorithm of the flow equation method is relative to the family of RG formulations.<sup>9</sup> Let us remark that such techniques are, in principle, unrestricted by limitations of the usual perturbative methods.

In this paper, we (1) formulate the current-current response function for the superconducting system using the flow equation method, and (2) extend such a scheme to a state of the preformed pairs, which, above the critical temperature  $T_c$ , lose the long-range coherence. Our study is motivated by the recent torque magnetometry data of the Princeton group,<sup>10</sup> revealing the diamagnetic features well preserved above  $T_c$  in the lanthanum and yttrium cuprate oxides. Similar indications have been also reported from the dc susceptibility measurements for  $\text{Bi}_{2.2}\text{Sr}_{1.8}\text{Ca}_2\text{Cu}_3\text{O}_{10+\delta}$  (Ref. 11) and from the high-resolution SQUID data for Sm-based underdoped YBCO compounds.<sup>12</sup> Since the observed diamagnetic response is rather strong, it can be hardly assigned to the Ginzburg-Landau fluctuations.<sup>13</sup>

Adopting argumentation discussed in the literature on the microscopic<sup>14–19</sup> and the phenomenological grounds,<sup>20,21</sup> we consider the system consisting of the preformed local pairs (of arbitrary origin) coexisting and interacting with the itinerant electrons. Using the flow equation approach, we analyze the diamagnetism within such a scenario. Our study can be

regarded as complementary to the recent quantum Monte Carlo (QMC) simulations for the same cooperon-fermion model.<sup>22</sup> It has also some resemblance to considerations of the superconducting fluctuations beyond the Gaussian approximation carried out in the  $t$ - $J$  model.<sup>23</sup>

We start by discussing the usual BCS model, treating it as a testing field for formulation of the linear response theory in terms of the flow equation method (readers less interested in the methodological details can skip this section). We next apply the same treatment to the mixture of electrons interacting through the Andreev scattering with the preformed local pairs. We determine the current-current response function and try to assess the diamagnetic response above  $T_c$ . In summary, we point out the main conclusions and give a list of problems that might be of interest for further studies.

## II. BCS SUPERCONDUCTOR

Let us first briefly illustrate how the flow equation procedure determines the quasiparticle spectrum and the corresponding response function for the usual BCS model

$$\hat{H} = \sum_{\mathbf{k}, \sigma} \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} (\Delta_{\mathbf{k}} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger + \text{H.c.}) \quad (1)$$

describing electrons coupled to the pairing field  $\Delta_{\mathbf{k}}$ . We use here the standard notation for the creation (annihilation) operators  $\hat{c}_{\mathbf{k}\sigma}^\dagger$  ( $\hat{c}_{\mathbf{k}\sigma}$ ) and denote by  $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$  the energies measured with respect to the chemical potential  $\mu$ . Formally,  $\Delta_{\mathbf{k}}$  can be thought of as the Bose-Einstein condensate of the Cooper pairs  $\Delta_{\mathbf{k}} = -\sum_{\mathbf{q}} V_{\mathbf{k}, \mathbf{q}} \langle \hat{c}_{-\mathbf{q}\downarrow} \hat{c}_{\mathbf{q}\uparrow} \rangle$ , which are formed by some attractive potential  $V_{\mathbf{k}, \mathbf{q}} < 0$ .

### A. Continuous diagonalization

The continuous diagonalization of the reduced BCS Hamiltonian (1) has been considered in the original paper by Wegner<sup>1</sup> and by several other authors.<sup>24,25</sup> We briefly recollect the main steps of such a procedure (see Appendix A for the procedural details), which shall be useful for formulating the linear response theory.

Following Wegner,<sup>1</sup> we choose the generating operator  $\hat{\eta}(l)$  defined by Eq. (A3), which, for the BCS model (1), has the following structure:

$$\hat{\eta}(l) = 2 \sum_{\mathbf{k}} \xi_{\mathbf{k}}(l) (\Delta_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow}^{\dagger} \hat{c}_{-\mathbf{k}\downarrow}^{\dagger} - \Delta_{\mathbf{k}}^*(l) \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow}). \quad (2)$$

Transformation of the Hamiltonian  $\hat{H}(l)$  proceeds as long as  $\eta(l)$  is finite, which occurs until  $\Delta_{\mathbf{k}}(l) \rightarrow 0$ . This is achieved in the asymptotic limit  $l \rightarrow \infty$  [Eq. (A4)].

By substituting (2) to the general flow equation (A2) for the Hamiltonian (1), we obtain<sup>25</sup>

$$\frac{d}{dl} \ln \xi_{\mathbf{k}}(l) = 4 |\Delta_{\mathbf{k}}(l)|^2, \quad (3)$$

$$\frac{d}{dl} \ln \Delta_{\mathbf{k}}(l) = -4 (\xi_{\mathbf{k}}(l))^2. \quad (4)$$

Equations (3) and (4) yield an exponential flow

$$\Delta_{\mathbf{k}}(l) = \Delta_{\mathbf{k}} e^{-4 \int_0^l dl' |\xi_{\mathbf{k}}(l')|^2} \quad (5)$$

and  $\xi_{\mathbf{k}}(l) = \xi_{\mathbf{k}} e^{4 \int_0^l dl' |\Delta_{\mathbf{k}}(l')|^2}$ ; therefore, the off-diagonal term  $\Delta_{\mathbf{k}}(l)$  vanishes in the limit  $l \rightarrow \infty$ . By combining Eqs. (3) and (4), we moreover notice that  $\frac{d}{dl} \{ \xi_{\mathbf{k}}^2(l) + |\Delta_{\mathbf{k}}(l)|^2 \} = 0$ , which implies the invariance  $\xi_{\mathbf{k}}^2(l) + |\Delta_{\mathbf{k}}(l)|^2 = \text{const}$ . Due to  $\Delta_{\mathbf{k}}(\infty) = 0$ , we conclude that the quasiparticle energies take the following BCS form:

$$\tilde{\xi}_{\mathbf{k}} = \text{sgn}(\xi_{\mathbf{k}}) \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}, \quad (6)$$

where we introduced the shorthand notation for the asymptotic value  $\tilde{\xi}_{\mathbf{k}} \equiv \lim_{l \rightarrow \infty} \xi_{\mathbf{k}}(l)$ .

### B. Single-particle excitations

As an illustration as to how one can use this procedure to obtain various Green's functions, let us derive the single-particle excitation spectrum determined by  $G_{\sigma}(\mathbf{k}, \tau) = -\hat{T}_{\tau} (\hat{c}_{\mathbf{k}\sigma}(\tau) \hat{c}_{\mathbf{k}\sigma}^{\dagger})$ , where  $\hat{T}_{\tau}$  denotes chronological ordering and  $\tau$  stands for the imaginary time. Applying (2) to the flow equation (A7) for the creation and annihilation operators, we infer the Bogoliubov ansatz

$$\hat{c}_{\mathbf{k}\uparrow}(l) = u_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}}(l) \hat{c}_{-\mathbf{k}\downarrow}^{\dagger}, \quad (7)$$

$$\hat{c}_{-\mathbf{k}\downarrow}^{\dagger}(l) = -v_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow} + u_{\mathbf{k}}(l) \hat{c}_{-\mathbf{k}\downarrow}^{\dagger}, \quad (8)$$

with the initial boundary conditions  $u_{\mathbf{k}}(0) = 1$ ,  $v_{\mathbf{k}}(0) = 0$ . By arranging the  $l$ -dependent coefficients in front of  $\hat{c}_{\mathbf{k}\uparrow}$  and  $\hat{c}_{-\mathbf{k}\downarrow}^{\dagger}$  on both sides of the flow equation (A7), we find that

$$\frac{du_{\mathbf{k}}(l)}{dl} = -2 \xi_{\mathbf{k}}(l) \Delta_{\mathbf{k}}^*(l) v_{\mathbf{k}}(l), \quad (9)$$

$$\frac{dv_{\mathbf{k}}(l)}{dl} = 2 \xi_{\mathbf{k}}(l) \Delta_{\mathbf{k}}(l) u_{\mathbf{k}}(l). \quad (10)$$

From Eqs. (9) and (10), we can see that the sum rule  $u_{\mathbf{k}}^2(l) + v_{\mathbf{k}}^2(l) = 1$  is properly conserved. To determine the needed asymptotic values, we can rewrite (9) as  $\frac{du_{\mathbf{k}}(l)}{v_{\mathbf{k}}(l)} = -2 \xi_{\mathbf{k}}(l) \Delta_{\mathbf{k}}(l) dl$ , and by substituting  $v_{\mathbf{k}}(l) = \sqrt{1 - u_{\mathbf{k}}^2(l)}$ , we

can analytically solve the integral  $\int_0^{\infty} dl$ . In the asymptotics, we obtain the usual Bogoliubov-Valatin coefficients

$$\tilde{u}_{\mathbf{k}}^2 = 1 - \tilde{v}_{\mathbf{k}}^2 = \frac{1}{2} \left[ 1 + \frac{\xi_{\mathbf{k}}}{\tilde{\xi}_{\mathbf{k}}} \right]. \quad (11)$$

Fourier transform of the single-particle Green's function

$$G_{\sigma}(\mathbf{k}, i\omega) = \beta^{-1} \int_0^{\beta} d\tau e^{-i\omega\tau} G_{\sigma}(\mathbf{k}, \tau) \quad (12)$$

(where  $\beta^{-1} = k_B T$ ) takes hence the two-pole structure

$$G_{\sigma}(\mathbf{k}, i\omega) = \frac{\tilde{u}_{\mathbf{k}}^2}{i\omega - \tilde{\xi}_{\mathbf{k}}} + \frac{\tilde{v}_{\mathbf{k}}^2}{i\omega + \tilde{\xi}_{\mathbf{k}}}, \quad (13)$$

signaling the particle-hole mixing, characteristic for the BCS state.

### C. Linear response theory

We now adopt the same procedure for studying a response of the BCS superconductor to a weak electromagnetic field  $\mathbf{A}(\mathbf{r}, t)$ . In the linear response, the induced current  $\mathbf{J}(\mathbf{r}, t)$  is assumed to be proportional to the perturbation, i.e.,  $\mathbf{J}(\mathbf{r}, t) = -\int d\mathbf{r}' \int_{-\infty}^t dt' K(\mathbf{r} - \mathbf{r}', t - t') \mathbf{A}(\mathbf{r}', t')$ . Fourier transform of the kernel function consists of the diamagnetic and paramagnetic contributions<sup>26</sup>

$$K_{\alpha, \beta}(\mathbf{q}, \omega) = \frac{ne^2}{m} \delta_{\alpha, \beta} + e^2 \Pi_{\alpha, \beta}(\mathbf{q}, \omega). \quad (14)$$

From now on, by  $\alpha, \beta$  we shall denote the Cartesian coordinates  $x, y$ , and  $z$ . The paramagnetic term can be expressed by the (analytically continued) Fourier transform (12) of the current-current Green's function

$$\Pi_{\alpha, \beta}(\mathbf{q}, \tau) \equiv -\langle \hat{T}_{\tau} \hat{j}_{\mathbf{q}, \alpha}(\tau) \hat{j}_{-\mathbf{q}, \beta} \rangle, \quad (15)$$

where the corresponding current operator  $\hat{\mathbf{j}}_{\mathbf{q}} = \hat{\mathbf{j}}_{\mathbf{q}}^{\uparrow} + \hat{\mathbf{j}}_{\mathbf{q}}^{\downarrow}$  consists of the spin-up ( $\uparrow$ ) and -down ( $\downarrow$ ) contributions

$$\hat{\mathbf{j}}_{\mathbf{q}}^{\sigma} = \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}+\frac{\mathbf{q}}{2}} \hat{c}_{\mathbf{k}, \sigma}^{\dagger} \hat{c}_{\mathbf{k}+\mathbf{q}, \sigma} \quad (16)$$

with the velocity  $\mathbf{v}_{\mathbf{k}} = \hbar^{-1} \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}}$ .

The standard way for computing the current-current response function (15) is based on the diagrammatic bubble-type contributions involving the particle and hole propagators and another contribution from the off-diagonal (in Nambu notation) single-particle propagators. In this section, we retrieve the standard BCS result<sup>26</sup> using the flow equation routine.

To guess the relevant flow of the current operators, we start by analyzing the initial ( $l = 0$ ) derivative

$$\left( \frac{d \hat{\mathbf{j}}_{\mathbf{q}}^{\sigma}(l)}{dl} \right)_{l=0} = [\hat{\eta}(l), \hat{\mathbf{j}}_{\mathbf{q}}^{\sigma}(l)]_{l=0}, \quad (17)$$

where  $\hat{\mathbf{j}}_{\mathbf{q}}^{\sigma}(l = 0)$  corresponds to the definition (16). Using the generating operator (2), we find that

$$\begin{aligned} \left( \frac{d \hat{\mathbf{j}}_{\mathbf{q}}^{\sigma}(l)}{dl} \right)_{l=0} &= \pm 2 \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}+\frac{\mathbf{q}}{2}} (\xi_{\mathbf{k}} \Delta_{\mathbf{k}} \hat{c}_{-\mathbf{k}, -\sigma} \hat{c}_{\mathbf{k}+\mathbf{q}, \sigma} \\ &\quad + \xi_{\mathbf{k}+\mathbf{q}} \Delta_{\mathbf{k}+\mathbf{q}} \hat{c}_{\mathbf{k}, \sigma}^{\dagger} \hat{c}_{-(\mathbf{k}+\mathbf{q}), -\sigma}), \end{aligned} \quad (18)$$

where the sign  $+$  ( $-$ ) refers to spin  $\uparrow$  ( $\downarrow$ ). Equation (18) unambiguously implies the following  $l$ -dependent parametrization:

$$\begin{aligned} \hat{\mathbf{j}}_q^\uparrow(l) &= \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}+\frac{\mathbf{q}}{2}} (\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{k},\uparrow}^\dagger \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow} + \mathcal{B}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{-(\mathbf{k}+\mathbf{q}),\downarrow}^\dagger) \\ &\quad + \mathcal{D}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{k},\uparrow}^\dagger \hat{c}_{-(\mathbf{k}+\mathbf{q}),\downarrow}^\dagger + \mathcal{F}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow}, \\ \hat{\mathbf{j}}_q^\downarrow(l) &= \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}+\frac{\mathbf{q}}{2}} (\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{k},\downarrow}^\dagger \hat{c}_{\mathbf{k}+\mathbf{q},\downarrow} + \mathcal{B}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{-\mathbf{k},\uparrow} \hat{c}_{-(\mathbf{k}+\mathbf{q}),\uparrow}^\dagger) \\ &\quad - \mathcal{F}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{k},\uparrow}^\dagger \hat{c}_{-(\mathbf{k}+\mathbf{q}),\downarrow}^\dagger - \mathcal{D}_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{-\mathbf{k},\downarrow} \hat{c}_{\mathbf{k}+\mathbf{q},\uparrow}, \end{aligned} \quad (19)$$

with the initial boundary conditions  $\mathcal{A}_{\mathbf{k},\mathbf{q}}(0) = 1$  and  $\mathcal{B}_{\mathbf{k},\mathbf{q}}(0) = \mathcal{D}_{\mathbf{k},\mathbf{q}}(0) = \mathcal{F}_{\mathbf{k},\mathbf{q}}(0) = 0$ . All  $l$ -dependent coefficients have to be determined by applying the ansatz (19) in the flow equation (A7) for the current operators  $\hat{\mathbf{j}}_q^\sigma(l)$ . On this basis, we obtain the following set of equations:

$$\frac{d\mathcal{A}_{\mathbf{k},\mathbf{q}}(l)}{dl} = -2[\xi_{\mathbf{k}+\mathbf{q}}(l)\Delta_{\mathbf{k}+\mathbf{q}}(l)\mathcal{D}_{\mathbf{k},\mathbf{q}}(l) + \xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)\mathcal{F}_{\mathbf{k},\mathbf{q}}(l)], \quad (20)$$

$$\frac{d\mathcal{B}_{\mathbf{k},\mathbf{q}}(l)}{dl} = 2[\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)\mathcal{D}_{\mathbf{k},\mathbf{q}}(l) + \xi_{\mathbf{k}+\mathbf{q}}(l)\Delta_{\mathbf{k}+\mathbf{q}}(l)\mathcal{F}_{\mathbf{k},\mathbf{q}}(l)], \quad (21)$$

$$\frac{d\mathcal{D}_{\mathbf{k},\mathbf{q}}(l)}{dl} = 2[\xi_{\mathbf{k}+\mathbf{q}}(l)\Delta_{\mathbf{k}+\mathbf{q}}(l)\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) - \xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)\mathcal{B}_{\mathbf{k},\mathbf{q}}(l)], \quad (22)$$

$$\frac{d\mathcal{F}_{\mathbf{k},\mathbf{q}}(l)}{dl} = 2[\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) - \xi_{\mathbf{k}+\mathbf{q}}(l)\Delta_{\mathbf{k}+\mathbf{q}}(l)\mathcal{B}_{\mathbf{k},\mathbf{q}}(l)]. \quad (23)$$

By inspecting Eqs. (20)–(23), we can notice that

$$\begin{aligned} \frac{d}{dl} [\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) + \mathcal{B}_{\mathbf{k},\mathbf{q}}(l)] &= 2[\xi_{\mathbf{k}+\mathbf{q}}(l)\Delta_{\mathbf{k}+\mathbf{q}}(l) - \xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)] \\ &\quad \times [\mathcal{F}_{\mathbf{k},\mathbf{q}}(l) - \mathcal{D}_{\mathbf{k},\mathbf{q}}(l)], \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{d}{dl} [\mathcal{F}_{\mathbf{k},\mathbf{q}}(l) - \mathcal{D}_{\mathbf{k},\mathbf{q}}(l)] &= -2[\xi_{\mathbf{k}+\mathbf{q}}(l)\Delta_{\mathbf{k}+\mathbf{q}}(l) - \xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)] \\ &\quad \times [\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) + \mathcal{B}_{\mathbf{k},\mathbf{q}}(l)], \end{aligned} \quad (25)$$

which implies

$$\frac{d}{dl} [\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) + \mathcal{B}_{\mathbf{k},\mathbf{q}}(l)]^2 + \frac{d}{dl} [\mathcal{D}_{\mathbf{k},\mathbf{q}}(l) - \mathcal{F}_{\mathbf{k},\mathbf{q}}(l)]^2 = 0. \quad (26)$$

Taking into account the initial boundary conditions, we hence obtain the following invariance:

$$[\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) + \mathcal{B}_{\mathbf{k},\mathbf{q}}(l)]^2 + [\mathcal{D}_{\mathbf{k},\mathbf{q}}(l) - \mathcal{F}_{\mathbf{k},\mathbf{q}}(l)]^2 = 1 \quad (27)$$

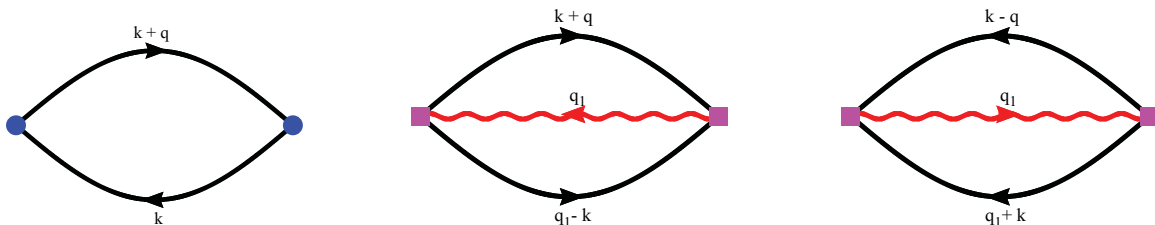


FIG. 1. (Color online) Contributions to the current-current response function from the particle-hole bubble (the left-hand-side panel) and from additional terms involving the finite-momentum boson propagators (the middle and right-hand-side panels). Vertices are expressed by the corresponding momentum components of the asymptotic values  $l \rightarrow \infty$  for the coefficients used in Eq. (37).

valid for arbitrary  $l$ , including the limit  $l \rightarrow \infty$ . Combining (27) with the differential equations (24) and (25), we exactly determine the asymptotic limit values

$$[\tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} + \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}}]^2 = \frac{1}{2} \left( 1 + \frac{\Delta_{\mathbf{k}+\mathbf{q}}\Delta_{\mathbf{k}} + \xi_{\mathbf{k}+\mathbf{q}}\xi_{\mathbf{k}}}{\tilde{\xi}_{\mathbf{k}+\mathbf{q}}\tilde{\xi}_{\mathbf{k}}} \right), \quad (28)$$

$$[\tilde{\mathcal{D}}_{\mathbf{k},\mathbf{q}} - \tilde{\mathcal{F}}_{\mathbf{k},\mathbf{q}}]^2 = \frac{1}{2} \left( 1 - \frac{\Delta_{\mathbf{k}+\mathbf{q}}\Delta_{\mathbf{k}} + \xi_{\mathbf{k}+\mathbf{q}}\xi_{\mathbf{k}}}{\tilde{\xi}_{\mathbf{k}+\mathbf{q}}\tilde{\xi}_{\mathbf{k}}} \right), \quad (29)$$

where  $\tilde{\xi}_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$ . In the same way, we also check that  $[\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) - \mathcal{B}_{\mathbf{k},\mathbf{q}}(l)]^2 - [\mathcal{D}_{\mathbf{k},\mathbf{q}}(l) + \mathcal{F}_{\mathbf{k},\mathbf{q}}(l)]^2 = 1$ , thereby the asymptotic values of all coefficients are found  $\tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} = \tilde{u}_{\mathbf{k}}\tilde{u}_{\mathbf{k}+\mathbf{q}}$ ,  $\tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}} = \tilde{v}_{\mathbf{k}}\tilde{v}_{\mathbf{k}+\mathbf{q}}$ ,  $\tilde{\mathcal{D}}_{\mathbf{k},\mathbf{q}} = \tilde{v}_{\mathbf{k}}\tilde{u}_{\mathbf{k}+\mathbf{q}}$ , and  $\tilde{\mathcal{F}}_{\mathbf{k},\mathbf{q}} = \tilde{u}_{\mathbf{k}}\tilde{v}_{\mathbf{k}+\mathbf{q}}$ .

Since the transformed Hamiltonian  $\hat{H}(\infty)$  is diagonal, we can easily compute the current-current response function (15) by expressing it through the particle-hole bubble diagrams (see the left-hand-side panel in Fig. 1). Finally, it is given by

$$\begin{aligned} \Pi_{\alpha,\beta}(\mathbf{q}, i\nu) &= \sum_{\mathbf{k}} v_{\mathbf{k}+\frac{\mathbf{q}}{2},\alpha} v_{\mathbf{k}+\frac{\mathbf{q}}{2},\beta} \left\{ [\tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} + \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}}]^2 [f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}}) \right. \\ &\quad - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})] \left[ \frac{1}{i\nu + \tilde{\xi}_{\mathbf{k}+\mathbf{q}} - \tilde{\xi}_{\mathbf{k}}} - \frac{1}{i\nu - \tilde{\xi}_{\mathbf{k}+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}}} \right] \\ &\quad + [\tilde{\mathcal{D}}_{\mathbf{k},\mathbf{q}} - \tilde{\mathcal{F}}_{\mathbf{k},\mathbf{q}}]^2 [1 - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}}) - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})] \\ &\quad \left. \times \left[ \frac{1}{i\nu - \tilde{\xi}_{\mathbf{k}+\mathbf{q}} - \tilde{\xi}_{\mathbf{k}}} - \frac{1}{i\nu + \tilde{\xi}_{\mathbf{k}+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}}} \right] \right\}, \end{aligned} \quad (30)$$

where  $f_{\text{FD}}(\omega) = [\exp(\omega/k_B T) + 1]^{-1}$  is the Fermi-Dirac distribution function. We recognize that Eqs. (28) and (29) correspond to the usual BCS coherence factors  $(\tilde{u}_{\mathbf{k}+\mathbf{q}}\tilde{u}_{\mathbf{k}} + \tilde{v}_{\mathbf{k}+\mathbf{q}}\tilde{v}_{\mathbf{k}})^2$  and  $(\tilde{u}_{\mathbf{k}+\mathbf{q}}\tilde{v}_{\mathbf{k}} - \tilde{v}_{\mathbf{k}+\mathbf{q}}\tilde{u}_{\mathbf{k}})^2$  and, thereby, Eq. (30) rigorously reproduces the known BCS response function.<sup>26</sup>

### III. SUPERCONDUCTING FLUCTUATIONS

Numerous experimental data<sup>10,27–35</sup> provided a rather clear evidence that the critical temperature  $T_c$  in the underdoped cuprate superconductors (and similarly in the ultracold fermion gases near the unitary limit<sup>36</sup>) is not related to appearance of the fermion pairs, but corresponds to the onset of their phase coherence. Upon approaching  $T_c$  from above, the short-range superconducting correlations gradually emerge. For instance, the torque magnetometry<sup>10</sup> and other measurements<sup>11,12</sup> have detected the diamagnetic properties.

To investigate the electrodynamic properties of the noncondensed preformed pairs, we consider the model

$$\hat{H} = \sum_{\mathbf{k}, \sigma} \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} E_{\mathbf{q}} \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}} + \frac{1}{\sqrt{N}} \sum_{\mathbf{k}, \mathbf{p}} g_{\mathbf{k}, \mathbf{p}} (\hat{b}_{\mathbf{k}+\mathbf{p}}^\dagger \hat{c}_{\mathbf{k}\downarrow} \hat{c}_{\mathbf{p}\uparrow} + \hat{b}_{\mathbf{k}+\mathbf{p}} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{p}\downarrow}^\dagger), \quad (31)$$

describing the itinerant electrons ( $\hat{c}_{\mathbf{k}\sigma}^{(\dagger)}$  operators) coexisting with the preformed pairs (bosonic  $\hat{b}_{\mathbf{q}}^{(\dagger)}$  operators). They are mutually coupled through the charge exchange (Andreev-type) scattering. Such scenario (31) has been considered by various authors in the context of high- $T_c$  superconductivity<sup>14–21</sup> and for description of the resonant Feshbach interaction in the ultracold fermion atom gases.<sup>36–38</sup>

By  $E_{\mathbf{q}}$ , we denote the energy of preformed pairs measured with respect to  $2\mu$ . Since in the superconducting state of cuprate materials the energy gap  $\Delta_{\mathbf{k}}$  has a  $d$ -wave symmetry, we furthermore impose the anisotropic boson-fermion coupling  $g_{\mathbf{k}, \mathbf{p}} = g(\cos k_x - \cos k_y)$ . If one restricts only to the Bose-Einstein condensed pairs (i.e., to bosonic  $\mathbf{q} = \mathbf{0}$  mode), then the model (31) becomes identical with the reduced BCS Hamiltonian (1), where  $\Delta_{\mathbf{k}}$  is related to the condensate  $-\frac{\hat{b}_{\mathbf{q}=\mathbf{0}}}{\sqrt{N}} g_{\mathbf{k}, -\mathbf{k}}$ . In what follows, we shall consider an influence of the noncondensed preformed pairs on the current-current response function.

### A. Outline of the continuous diagonalization

Adopting again Wegner's proposal,<sup>1</sup> we choose the generating operator as  $\hat{\eta}(l) = [\hat{H}_0(l), \hat{V}^{B-F}(l)]$ , where  $\hat{H}_0(l)$  stands for the free fermion and boson contributions, whereas  $\hat{V}^{B-F}(l)$  denotes their interaction term. In explicit form, such a generating operator is given by

$$\hat{\eta}(l) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}, \mathbf{p}} \alpha_{\mathbf{k}, \mathbf{p}}(l) (\hat{b}_{\mathbf{k}+\mathbf{p}} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{p}\downarrow}^\dagger - \text{H.c.}), \quad (32)$$

where  $\alpha_{\mathbf{k}, \mathbf{p}}(l) = [\xi_{\mathbf{k}}(l) + \xi_{\mathbf{p}}(l) - E_{\mathbf{k}+\mathbf{p}}(l)] g_{\mathbf{k}, \mathbf{p}}(l)$ . By using (32) in the flow equation for the Hamiltonian (31), we obtain<sup>39</sup>

$$\frac{d}{dl} \ln g_{\mathbf{k}, \mathbf{p}}(l) = -[\xi_{\mathbf{k}}(l) + \xi_{\mathbf{p}}(l) - E_{\mathbf{k}+\mathbf{p}}(l)]^2, \quad (33)$$

which implies an exponential diminishing of  $g_{\mathbf{k}, \mathbf{p}}(l)$  and guarantees its total disappearance in the asymptotic limit  $l \rightarrow \infty$ . Simultaneously, the fermion and boson energies are renormalized according to the flow equations<sup>39</sup>

$$\frac{d}{dl} \xi_{\mathbf{k}}(l) = \frac{2}{N} \sum_{\mathbf{q}} \alpha_{\mathbf{k}, \mathbf{q}-\mathbf{k}}(l) g_{\mathbf{k}, \mathbf{q}-\mathbf{k}}(l) n_{\mathbf{q}}^B, \quad (34)$$

$$\frac{d}{dl} E_{\mathbf{q}}(l) = -\frac{2}{N} \sum_{\mathbf{k}} \alpha_{\mathbf{k}, \mathbf{k}-\mathbf{q}}(l) g_{\mathbf{k}-\mathbf{q}, \mathbf{k}}(l) [-n_{\mathbf{k}-\mathbf{q}}^F - n_{\mathbf{k}}^F], \quad (35)$$

where  $n_{\mathbf{k}}^{F,B}$  denote the fermion and boson occupancies of momentum  $\mathbf{k}$  state. We have previously<sup>39,40</sup> explored (analytically and numerically) the flow equations (33)–(35), arriving at the following conclusions:

(i) The renormalized fermion dispersion  $\tilde{\xi}_{\mathbf{k}}$  develops either the true gap (below  $T_c$ , when a finite fraction of the Bose-Einstein (BE) condensed bosons exists) or the pseudogap (for  $T_c < T < T^*$ , where  $T^*$  marks the onset of superconducting-type correlations).

(ii) The long-wavelength limit of the effective boson dispersion  $\tilde{E}_{\mathbf{q}}$  is characterized by the Goldstone mode (for  $T < T_c$ ), the remnants of which become overdamped in the pseudogap regime (above  $T_c$ ),

(iii) The single-particle spectral function of fermions (see Appendix B) consists of the Bogoliubov-type branches separated by the (pseudo)gap, and these features remain preserved up to  $T^*$ .

More recently,<sup>19</sup> we have also investigated evolution of the  $\mathbf{k}$ -resolved pseudogap considering two-dimensional lattice dispersion with the nearest- and next-nearest-neighbor hopping integrals realistic for the cuprate superconductors. For temperatures slightly above  $T_c$ , we have found that the pseudogap starts to close around the nodal points restoring the Fermi arcs, whereas in the antinodal areas, the pseudogap practically does not change. Upon a gradual increase of temperature, the length of the Fermi arcs linearly increases, in agreement with the experimental angle-resolved photoemission spectroscopy (ARPES) data.<sup>41</sup> Similar conclusions have been achieved for the same model (31) from theoretical studies based on the conserving diagrammatic approach.<sup>42</sup>

### B. Diamagnetism due to the preexisting pairs

Following the guidelines discussed in Sec. II C, we can now formulate the linear response theory for the model (31), focusing on the role played by the noncondensed  $\mathbf{q} \neq \mathbf{0}$  preformed pairs.

To impose the corresponding parametrization of the current operators  $\mathbf{j}_{\mathbf{q}}^{\sigma}$ , we again start from the initial derivative (17). By using the generating operator (32), we obtain

$$\left( \frac{d \hat{\mathbf{j}}_{\mathbf{q}}^{\sigma}(l)}{dl} \right)_{l=0} = \mp \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}+\frac{\mathbf{q}}{2}} \sum_{\mathbf{p}} (\alpha_{\mathbf{k}, \mathbf{p}} \hat{b}_{\mathbf{k}+\mathbf{p}}^\dagger \hat{c}_{\mathbf{k}+\mathbf{p}, -\sigma} \hat{c}_{\mathbf{k}+\mathbf{q}, \sigma} + \alpha_{\mathbf{k}+\mathbf{q}, \mathbf{k}} \hat{b}_{\mathbf{k}+\mathbf{q}+\mathbf{p}} \hat{c}_{\mathbf{k}, \sigma}^\dagger \hat{c}_{\mathbf{p}, -\sigma}^\dagger), \quad (36)$$

where  $- (+)$  refers to the spin  $\uparrow (\downarrow)$ . By analyzing (36), we deduce the following general structure of the  $l$ -dependent current operators:

$$\hat{\mathbf{j}}_{\mathbf{q}}^{\uparrow}(l) = \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}+\frac{\mathbf{q}}{2}} [\mathcal{A}_{\mathbf{k}, \mathbf{q}}(l) \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{k}+\mathbf{q}, \uparrow} + \mathcal{B}_{\mathbf{k}, \mathbf{q}}(l) \hat{c}_{-\mathbf{k}, \downarrow} \hat{c}_{-(\mathbf{k}+\mathbf{q}), \downarrow}^\dagger + \sum_{\mathbf{p}} (\mathcal{D}_{\mathbf{k}, \mathbf{p}, \mathbf{q}}(l) \hat{b}_{\mathbf{k}+\mathbf{p}} \hat{c}_{\mathbf{k}, \uparrow}^\dagger \hat{c}_{\mathbf{p}-\mathbf{q}, \downarrow}^\dagger + \mathcal{F}_{\mathbf{k}, \mathbf{p}, \mathbf{q}}(l) \hat{b}_{\mathbf{k}+\mathbf{p}}^\dagger \hat{c}_{\mathbf{p}, \downarrow} \hat{c}_{\mathbf{k}+\mathbf{q}, \uparrow})] \quad (37)$$

with the initial values  $\mathcal{A}_{\mathbf{k}, \mathbf{q}}(0) = 1$  and  $\mathcal{B}_{\mathbf{k}, \mathbf{q}}(0) = \mathcal{D}_{\mathbf{k}, \mathbf{p}, \mathbf{q}}(0) = \mathcal{F}_{\mathbf{k}, \mathbf{p}, \mathbf{q}}(0) = 0$ . The operator  $\hat{\mathbf{j}}_{\mathbf{q}}^{\downarrow}(l)$  is given by the expression analogous to (37) with  $\mathcal{D}_{\mathbf{k}, \mathbf{p}, \mathbf{q}}(l)$  replaced by  $-\mathcal{F}_{\mathbf{k}, \mathbf{p}, \mathbf{q}}(l)$  and *vice versa*. Let us remark that by taking into account only the BE condensed pairs  $\hat{b}_{\mathbf{k}+\mathbf{p}}^{(\dagger)} = \hat{b}_0^{(\dagger)} \delta_{\mathbf{p}, -\mathbf{k}}$ , we would come back to the ansatz (19) reproducing the BCS solution.

After a somewhat lengthy but rather straightforward algebra, we derive the following set of the flow equations:

$$\frac{d\mathcal{A}_{\mathbf{k},\mathbf{q}}(l)}{dl} = \sum_{\mathbf{p}} [\alpha_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}(l)\mathcal{D}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l)(n_{\mathbf{p}-\mathbf{q}}^F + n_{\mathbf{k}+\mathbf{p}}^B) + \alpha_{\mathbf{k},\mathbf{p}}(l)\mathcal{F}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l)(n_{\mathbf{p}}^F + n_{\mathbf{k}+\mathbf{p}}^B)], \quad (38)$$

$$\frac{d\mathcal{B}_{\mathbf{k},\mathbf{q}}(l)}{dl} = - \sum_{\mathbf{p}} [\alpha_{\mathbf{k},\mathbf{p}}(l)\mathcal{D}_{-\mathbf{p},-\mathbf{k},\mathbf{q}}(l)(n_{\mathbf{p}}^F + n_{\mathbf{k}+\mathbf{p}}^B) + \alpha_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}(l)\mathcal{F}_{-\mathbf{p},-\mathbf{k},\mathbf{q}}(l)(n_{\mathbf{p}-\mathbf{q}}^F + n_{\mathbf{k}+\mathbf{p}}^B)], \quad (39)$$

$$\frac{d\mathcal{D}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l)}{dl} = -\alpha_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}(l)\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) + \alpha_{\mathbf{k},\mathbf{p}}(l)\mathcal{B}_{-\mathbf{p},\mathbf{q}}(l), \quad (40)$$

$$\frac{d\mathcal{F}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l)}{dl} = -\alpha_{\mathbf{k},\mathbf{p}}(l)\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) + \alpha_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}(l)\mathcal{B}_{-\mathbf{p},\mathbf{q}}(l). \quad (41)$$

For deriving the equations (38) and (39), we used the following approximations:

$$\begin{aligned} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}'} \hat{c}_{\mathbf{p},\sigma}^\dagger \hat{c}_{\mathbf{p}',\sigma} &\simeq \delta_{\mathbf{k},\mathbf{k}'} n_{\mathbf{k}}^B \hat{c}_{\mathbf{p},\sigma}^\dagger \hat{c}_{\mathbf{p}',\sigma} + \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}'} \delta_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}}^F \\ &- \delta_{\mathbf{k},\mathbf{k}'} n_{\mathbf{k}}^B \delta_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}}^F, \end{aligned} \quad (42)$$

$$\begin{aligned} \hat{c}_{\mathbf{p},\uparrow}^\dagger \hat{c}_{\mathbf{p}',\uparrow} \hat{c}_{\mathbf{p},\downarrow}^\dagger \hat{c}_{\mathbf{p}',\downarrow} &\simeq \delta_{\mathbf{k},\mathbf{k}'} n_{\mathbf{k}}^F \hat{c}_{\mathbf{p},\downarrow}^\dagger \hat{c}_{\mathbf{p}',\downarrow} + \hat{c}_{\mathbf{p},\uparrow}^\dagger \hat{c}_{\mathbf{p}',\uparrow} \delta_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}}^F \\ &- \delta_{\mathbf{k},\mathbf{k}'} n_{\mathbf{k}}^F \delta_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}}^F, \end{aligned} \quad (43)$$

neglecting the higher-order products  $\delta\hat{X}\delta\hat{Y}$  of the fluctuations  $\delta\hat{X} = \hat{X} - \langle\hat{X}\rangle$ , where the corresponding observables for the case of Eq. (42) are defined by  $\hat{X} = \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}'}$ ,  $\hat{Y} = \hat{c}_{\mathbf{p},\sigma}^\dagger \hat{c}_{\mathbf{p}',\sigma}$  and for (43) by  $\hat{X} = \hat{c}_{\mathbf{p},\uparrow}^\dagger \hat{c}_{\mathbf{p}',\uparrow}$ ,  $\hat{Y} = \hat{c}_{\mathbf{p},\downarrow}^\dagger \hat{c}_{\mathbf{p}',\downarrow}$ . Such truncations [Eqs. (42) and (43)] enable us to satisfy the flow equation  $\frac{d}{dl} \hat{\mathbf{j}}_{\mathbf{q}}^\sigma(l) = [\hat{\eta}(l), \hat{\mathbf{j}}_{\mathbf{q}}^\sigma(l)]$  using the parametrization (37) imposed on the current operators  $\hat{\mathbf{j}}_{\mathbf{q}}^\sigma(l)$ . Otherwise, if we introduced these neglected terms  $\hat{X}\hat{Y}$  to the  $l$ -dependent operator  $\hat{\mathbf{j}}_{\mathbf{q}}^\sigma(l)$ , they would induce even more complex structures arising from the commutator  $[\hat{\eta}(l), \hat{\mathbf{j}}_{\mathbf{q}}^\sigma(l)]$ , and formally the flow equation (A7) could never be obeyed (except for only the exactly solvable cases). The truncations (42) and (43), or similar, represent thus a necessary compromise in which the flow equation technique deals with the physical problems that are not exactly solvable.<sup>1,2</sup>

Finally, let us determine the current-current response function (15), keeping in mind that the statistical averaging is feasible with respect to  $\hat{H}(\infty)$ . Using the ansatz (37), we find the response function

$$\begin{aligned} &\langle\langle \hat{\mathbf{j}}_{\mathbf{q},\alpha}; \hat{\mathbf{j}}_{-\mathbf{q},\beta} \rangle\rangle \\ &= \sum_{\mathbf{k},\mathbf{p}} v_{\mathbf{k}+\frac{\mathbf{q}}{2},\alpha} v_{\mathbf{p}-\frac{\mathbf{q}}{2},\beta} \left( \tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} \tilde{\mathcal{A}}_{\mathbf{p},-\mathbf{q}} \sum_{\sigma} \langle\langle \hat{c}_{\mathbf{k},\sigma}^\dagger \hat{c}_{\mathbf{k}+\mathbf{q},\sigma}; \hat{c}_{\mathbf{p},\sigma}^\dagger \hat{c}_{\mathbf{p}-\mathbf{q},\sigma} \rangle\rangle + \tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} \tilde{\mathcal{B}}_{\mathbf{p},-\mathbf{q}} \sum_{\sigma} \langle\langle \hat{c}_{\mathbf{k},\sigma}^\dagger \hat{c}_{\mathbf{k}+\mathbf{q},\sigma}; \hat{c}_{-\mathbf{p},\sigma}^\dagger \hat{c}_{-(\mathbf{p}-\mathbf{q}),\sigma} \rangle\rangle \right. \\ &+ \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}} \tilde{\mathcal{A}}_{\mathbf{p},-\mathbf{q}} \sum_{\sigma} \langle\langle \hat{c}_{-\mathbf{k},\sigma}^\dagger \hat{c}_{-(\mathbf{k}+\mathbf{q}),\sigma}; \hat{c}_{\mathbf{p},\sigma}^\dagger \hat{c}_{\mathbf{p}-\mathbf{q},\sigma} \rangle\rangle + \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}} \tilde{\mathcal{B}}_{\mathbf{p},-\mathbf{q}} \sum_{\sigma} \langle\langle \hat{c}_{-\mathbf{k},\sigma}^\dagger \hat{c}_{-(\mathbf{k}+\mathbf{q}),\sigma}; \hat{c}_{-\mathbf{p},\sigma}^\dagger \hat{c}_{-(\mathbf{p}-\mathbf{q}),\sigma} \rangle\rangle \\ &\left. - \sum_{\mathbf{k}',\mathbf{p}'} \tilde{\mathcal{G}}_{\mathbf{k},\mathbf{k}',\mathbf{q}} \tilde{\mathcal{G}}_{\mathbf{p},\mathbf{p}',-\mathbf{q}} \langle\langle \hat{b}_{\mathbf{k}+\mathbf{k}'} \hat{c}_{\mathbf{k}',\uparrow}^\dagger \hat{c}_{\mathbf{k}'-\mathbf{q},\downarrow}; \hat{b}_{\mathbf{p}+\mathbf{p}'}^\dagger \hat{c}_{\mathbf{p}',\downarrow} \hat{c}_{\mathbf{p}-\mathbf{q},\uparrow} \rangle\rangle - \sum_{\mathbf{k}',\mathbf{p}'} \tilde{\mathcal{G}}_{\mathbf{p},\mathbf{p}',\mathbf{q}} \tilde{\mathcal{G}}_{\mathbf{k},\mathbf{k}',-\mathbf{q}} \langle\langle \hat{b}_{\mathbf{p}+\mathbf{p}'}^\dagger \hat{c}_{\mathbf{p}',\downarrow} \hat{c}_{\mathbf{p}+\mathbf{q},\uparrow}; \hat{b}_{\mathbf{p}+\mathbf{p}'}^\dagger \hat{c}_{\mathbf{p}',\downarrow} \hat{c}_{\mathbf{p}+\mathbf{q},\uparrow} \rangle\rangle \right), \quad (44) \end{aligned}$$

where  $\tilde{\mathcal{G}}_{\mathbf{p},\mathbf{p}',\mathbf{q}} \equiv \tilde{\mathcal{D}}_{\mathbf{p},\mathbf{p}',\mathbf{q}} - \tilde{\mathcal{F}}_{\mathbf{p},\mathbf{p}',\mathbf{q}}$  and we used the abbreviation  $\langle\langle \hat{O}_1; \hat{O}_2 \rangle\rangle \equiv -\langle \hat{T}_\tau \hat{O}_1(\tau) \hat{O}_2 \rangle_{\hat{H}(\infty)}$ . These contributions (44) are depicted graphically in Fig. 1. Vertices denoted by the filled squares correspond to the asymptotic value  $\tilde{\mathcal{G}}$ , whereas the filled circles represent  $\tilde{\mathcal{A}}$  and/or  $\tilde{\mathcal{B}}$ .

Performing the Matsubara summation for the particle-hole convolutions (left panel in Fig. 1) and the double Matsubara summation for the diagrams involving one bosonic and two fermionic propagators, we obtain the following Fourier transform of (44):

$$\begin{aligned} \Pi_{\alpha,\beta}(\mathbf{q},i\nu) &= \sum_{\mathbf{k}} v_{\mathbf{k}+\frac{\mathbf{q}}{2},\alpha} v_{\mathbf{k}+\frac{\mathbf{q}}{2},\beta} \left\{ [\tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} \tilde{\mathcal{A}}_{\mathbf{k}+\mathbf{q},-\mathbf{q}} + \tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} \tilde{\mathcal{B}}_{-\mathbf{k},-\mathbf{q}} \right. \\ &+ \tilde{\mathcal{A}}_{-\mathbf{k},-\mathbf{q}} \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}} + \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}} \tilde{\mathcal{B}}_{\mathbf{k}+\mathbf{q},-\mathbf{q}}] [f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}}) - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})] \left[ \frac{1}{i\nu + \tilde{\xi}_{\mathbf{k}+\mathbf{q}} - \tilde{\xi}_{\mathbf{k}}} - \frac{1}{i\nu - \tilde{\xi}_{\mathbf{k}+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}}} \right] \\ &+ \sum_{\mathbf{k}'} \tilde{\mathcal{G}}_{\mathbf{k},-\mathbf{k}',\mathbf{q}} \tilde{\mathcal{G}}_{\mathbf{k}+\mathbf{q},-(\mathbf{k}'+\mathbf{q}),-\mathbf{q}} \left( [1 - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}}) - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})] \frac{f_{\text{BE}}(\tilde{E}_{\mathbf{k}-\mathbf{k}'}) - f_{\text{BE}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}'})}{i\nu - (\tilde{\xi}_{\mathbf{k}+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}'} - \tilde{E}_{\mathbf{k}-\mathbf{k}'})} \right. \\ &\left. - [1 - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}'+\mathbf{q}}) - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})] \frac{f_{\text{BE}}(\tilde{E}_{\mathbf{k}-\mathbf{k}'}) - f_{\text{BE}}(\tilde{\xi}_{\mathbf{k}'+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}})}{i\nu + (\tilde{\xi}_{\mathbf{k}'+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}} - \tilde{E}_{\mathbf{k}-\mathbf{k}'})} \right) \left. \right\}, \quad (45) \end{aligned}$$

where  $f_{\text{BE}}(\omega) = [\exp(\omega/k_B T) - 1]^{-1}$  is the Bose-Einstein distribution. The function (45) in a straightforward manner generalizes the previous BCS form (30) and is the central result of our study. The dc diamagnetic properties of the system depend on the

static value of the response function. In our present case, it is given by

$$\begin{aligned} \Pi_{\alpha,\alpha}(\mathbf{q}, i\nu = 0) = & \sum_{\mathbf{k}} v_{\mathbf{k}+\frac{\mathbf{q}}{2},\alpha}^2 \left\{ 2[\tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}}\tilde{\mathcal{A}}_{\mathbf{k}+\mathbf{q},-\mathbf{q}} + \tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}}\tilde{\mathcal{B}}_{-\mathbf{k},-\mathbf{q}} + \tilde{\mathcal{A}}_{-\mathbf{k},-\mathbf{q}}\tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}} + \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}}\tilde{\mathcal{B}}_{\mathbf{k}+\mathbf{q},-\mathbf{q}}] \frac{f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}}) - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})}{\tilde{\xi}_{\mathbf{k}+\mathbf{q}} - \tilde{\xi}_{\mathbf{k}}} \right. \\ & + \sum_{\mathbf{k}'} \tilde{\mathcal{G}}_{\mathbf{k},-\mathbf{k}',\mathbf{q}} \tilde{\mathcal{G}}_{\mathbf{k}+\mathbf{q},-(\mathbf{k}'+\mathbf{q}),-\mathbf{q}} \left( [f_{\text{BE}}(\tilde{E}_{\mathbf{k}-\mathbf{k}'} - f_{\text{BE}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}'})] \frac{1 - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}+\mathbf{q}}) - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}'})}{\tilde{E}_{\mathbf{k}-\mathbf{k}'} - (\tilde{\xi}_{\mathbf{k}+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}'})} \right. \\ & \left. \left. + [f_{\text{BE}}(\tilde{E}_{\mathbf{k}-\mathbf{k}'} - f_{\text{BE}}(\tilde{\xi}_{\mathbf{k}'+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}})] \frac{1 - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}'+\mathbf{q}}) - f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})}{\tilde{E}_{\mathbf{k}-\mathbf{k}'} - (\tilde{\xi}_{\mathbf{k}'+\mathbf{q}} + \tilde{\xi}_{\mathbf{k}})} \right) \right\}. \end{aligned} \quad (46)$$

For temperatures below  $T_c$  (when a finite fraction of the Bose-Einstein condensed pairs exists), the main contribution in the expression (46) comes from  $\mathbf{k}' = \mathbf{k}$  terms. Under such conditions, Eq. (46) becomes identical with the BCS solution, consisting of (a) the *superfluid* fraction [i.e., the term in Eq. (30) proportional to the coherence factor  $(\tilde{u}_{\mathbf{k}+\mathbf{q}}\tilde{v}_{\mathbf{k}} - \tilde{v}_{\mathbf{k}+\mathbf{q}}\tilde{u}_{\mathbf{k}})^2$ ] and the other (b) *normal* contribution from the thermally excited quasiparticles, i.e., the term proportional to  $(\tilde{u}_{\mathbf{k}+\mathbf{q}}\tilde{u}_{\mathbf{k}} + \tilde{v}_{\mathbf{k}+\mathbf{q}}\tilde{v}_{\mathbf{k}})^2$ .<sup>43</sup>

Above  $T_c$  (but fairly below  $T^*$ ), a considerable amount of the preformed pairs occupies the low-momenta states  $E_{\mathbf{q}\rightarrow 0}$ , therefore, expression (46) becomes reminiscent of the above-mentioned BCS components in the response function.

#### IV. ITERATIVE SOLUTION

To explore the physical aspects related to the current-current response function (45), we adopt an iterative method for solving the coupled flow equations (38)–(41). Such a scheme allows for an approximate estimation of the introduced  $l$ -dependent parameters. In the following,<sup>39</sup> we make use of the fact that the dominant renormalization affects the boson-fermion coupling  $g_{\mathbf{k},\mathbf{p}}(l)$ , which ultimately vanishes in the asymptotic limit  $l \rightarrow \infty$ . By neglecting the simultaneous renormalization of the fermion  $\xi_{\mathbf{k}}(l) \simeq \xi_{\mathbf{k}}$  and boson energies  $E_{\mathbf{q}}(l) \simeq E_{\mathbf{q}}$ , we obtain the following solution of the flow equation (33):

$$g_{\mathbf{k},\mathbf{p}}(l) \simeq g_{\mathbf{k},\mathbf{p}} e^{-(\xi_{\mathbf{k}} + \xi_{\mathbf{p}} - E_{\mathbf{k}+\mathbf{p}})^2 l}. \quad (47)$$

By substituting this result (47) to the flow equations (34) and (35), we might in turn update the energies and the routine can be continued at each iterative level providing a better and better estimation for the renormalized quantities.

In this section, we apply such a scheme to the flow equations (38)–(41), restricting ourselves to the lowest-order solutions based on Eq. (47). We start with the initial values of the coefficients  $\mathcal{A}_{\mathbf{k},\mathbf{q}}(l) \simeq \mathcal{A}_{\mathbf{k},\mathbf{q}}(0) = 1$  and  $\mathcal{B}_{\mathbf{k},\mathbf{q}}(l) \simeq \mathcal{B}_{\mathbf{k},\mathbf{q}}(0) = 0$  substituting them in the right-hand side of the flow equations (40) and (41). Using Eq. (47), we analytically solve the simplified equations (40) and (41), obtaining

$$\mathcal{D}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l) \simeq \frac{g_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}} [e^{-(\xi_{\mathbf{k}+\mathbf{q}} + \xi_{\mathbf{p}-\mathbf{q}} - E_{\mathbf{k}+\mathbf{p}})^2 l} - 1]}{\xi_{\mathbf{k}+\mathbf{q}} + \xi_{\mathbf{p}-\mathbf{q}} - E_{\mathbf{k}+\mathbf{p}}}, \quad (48)$$

$$\mathcal{F}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l) \simeq \frac{g_{\mathbf{k},\mathbf{p}} [e^{-(\xi_{\mathbf{k}} + \xi_{\mathbf{p}} - E_{\mathbf{k}+\mathbf{p}})^2 l} - 1]}{\xi_{\mathbf{k}} + \xi_{\mathbf{p}} - E_{\mathbf{k}+\mathbf{p}}}. \quad (49)$$

Their asymptotic values are given by

$$\tilde{\mathcal{D}}_{\mathbf{k},\mathbf{p},\mathbf{q}} \simeq - \frac{g_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}}{\xi_{\mathbf{k}+\mathbf{q}} + \xi_{\mathbf{p}-\mathbf{q}} - E_{\mathbf{k}+\mathbf{p}}}, \quad (50)$$

$$\tilde{\mathcal{F}}_{\mathbf{k},\mathbf{p},\mathbf{q}} \simeq - \frac{g_{\mathbf{k},\mathbf{p}}}{\xi_{\mathbf{k}} + \xi_{\mathbf{p}} - E_{\mathbf{k}+\mathbf{p}}}. \quad (51)$$

By using the  $l$ -dependent coefficients  $\mathcal{D}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l)$  and  $\mathcal{F}_{\mathbf{k},\mathbf{p},\mathbf{q}}(l)$ , we can next determine  $\mathcal{A}_{\mathbf{k},\mathbf{p}}(l)$  and  $\mathcal{B}_{\mathbf{k},\mathbf{p}}(l)$ . By substituting Eqs. (48) and (49) in the right-hand side of (38) and (39), we obtain the following asymptotic values:

$$\begin{aligned} \tilde{\mathcal{A}}_{\mathbf{k},\mathbf{q}} \simeq & 1 - \frac{1}{2} \sum_{\mathbf{p}} \left[ \frac{(n_{\mathbf{p}}^F + n_{\mathbf{k}+\mathbf{p}}^B) |g_{\mathbf{k},\mathbf{p}}|^2}{(\xi_{\mathbf{k}} + \xi_{\mathbf{p}} - E_{\mathbf{k}+\mathbf{p}})^2} \right. \\ & \left. + \frac{(n_{\mathbf{p}-\mathbf{q}}^F + n_{\mathbf{k}+\mathbf{p}}^B) |g_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}|^2}{(\xi_{\mathbf{k}+\mathbf{q}} + \xi_{\mathbf{p}-\mathbf{q}} - E_{\mathbf{k}+\mathbf{p}})^2} \right] \end{aligned} \quad (52)$$

and

$$\begin{aligned} \tilde{\mathcal{B}}_{\mathbf{k},\mathbf{q}} \simeq & \sum_{\mathbf{p}} g_{\mathbf{k},\mathbf{p}} g_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}} \\ & \times \left[ \frac{n_{\mathbf{p}}^F + n_{\mathbf{k}+\mathbf{p}}^B}{X_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}} \left( \frac{1}{X_{\mathbf{k},\mathbf{p}}} - \frac{X_{\mathbf{k},\mathbf{p}}}{X_{\mathbf{k},\mathbf{p}}^2 + X_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}^2} \right) \right. \\ & \left. + \frac{n_{\mathbf{p}-\mathbf{q}}^F + n_{\mathbf{k}+\mathbf{p}}^B}{X_{\mathbf{k},\mathbf{p}}} \left( \frac{1}{X_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}} - \frac{X_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}}{X_{\mathbf{k},\mathbf{p}}^2 + X_{\mathbf{k}+\mathbf{q},\mathbf{p}-\mathbf{q}}^2} \right) \right], \end{aligned} \quad (53)$$

where  $X_{\mathbf{k},\mathbf{p}} \equiv \xi_{\mathbf{k}} + \xi_{\mathbf{p}} - E_{\mathbf{k}+\mathbf{p}}$ .

Since eventual diamagnetism is determined by the long-wavelength limit of the static response function (46), we focus on  $\mathbf{q} = \mathbf{0}$  values of the coefficients. By examining the  $\mathbf{q} \rightarrow \mathbf{0}$  limit of the asymptotic values (50) and (51), we notice that the *superfluid* vertices vanish:

$$\tilde{\mathcal{D}}_{\mathbf{k},\mathbf{p},\mathbf{q}} = \tilde{\mathcal{D}}_{\mathbf{k},\mathbf{p},\mathbf{q}} - \tilde{\mathcal{F}}_{\mathbf{k},\mathbf{p},\mathbf{q}} \xrightarrow{\mathbf{q}=\mathbf{0}} 0 \quad (54)$$

and (similarly to the BCS treatment<sup>43</sup>) we are left only with the *normal* component of the paramagnetic term

$$\lim_{\mathbf{q} \rightarrow \mathbf{0}} \Pi_{\alpha,\alpha}(\mathbf{q}, 0) \simeq 2 \sum_{\mathbf{k}} v_{\mathbf{k},\alpha}^2 [\mathcal{A}_{\mathbf{k},\mathbf{0}} + \mathcal{B}_{\mathbf{k},\mathbf{0}}]^2 \frac{df_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})}{d\tilde{\xi}_{\mathbf{k}}}. \quad (55)$$

At high temperatures (in a normal state), the dispersion  $\tilde{\xi}_{\mathbf{k}}$  is nearly identical with the bare energy  $\varepsilon_{\mathbf{k}} - \mu$ ; therefore, Eq. (55) cancels out the diamagnetic term of the response kernel  $K_{\alpha,\beta}(\mathbf{q} \rightarrow \mathbf{0}, 0)$  and, consequently, the system does not show any diamagnetic features. On the other hand, in

the superconducting state, the single-particle excitations  $\tilde{\xi}_{\mathbf{k}}$  are gapped, and at low temperatures,  $\frac{d}{d\tilde{\xi}_{\mathbf{k}}} f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}}) \approx -\delta(\tilde{\xi}_{\mathbf{k}})$ ; therefore, the paramagnetic contribution (55) vanishes.<sup>43</sup> One then obtains a perfect diamagnetism with the characteristic London penetration depth  $\lambda_L^{-2} = ne^2/m$ . Between these extreme regimes, we can expect an intermediate behavior. In particular, for temperatures  $T_c < T < T^*$ , the single-particle fermion spectrum becomes partly depleted around the Fermi energy, so the paramagnetic term (55) would no longer be able to compensate completely the diamagnetic contribution generating a fragile diamagnetism.

For some quantitative illustration of this behavior, we have analyzed temperature dependence of the superfluid fraction  $n_s(T)$  defined by the relation<sup>26</sup>

$$J_x(\mathbf{q} \rightarrow \mathbf{0}, 0) = -\frac{e^2 n_s(T)}{m} A_x(\mathbf{q} \rightarrow \mathbf{0}, 0). \quad (56)$$

In order to determine  $n_s(T)$ , we substituted the paramagnetic term (55) to the kernel function  $K_{\alpha,\beta}(\mathbf{q} \rightarrow \mathbf{0}, 0)$  and applied the coefficients (52) and (53), simplifying the fermion and boson concentrations by  $n_{\mathbf{k}}^F \approx f_{\text{FD}}(\tilde{\xi}_{\mathbf{k}})$  and  $n_{\mathbf{q}}^B \approx f_{\text{BE}}(\tilde{E}_{\mathbf{q}})$ . Furthermore, we replaced all energies by the renormalized values  $\tilde{\xi}_{\mathbf{k}}$  and  $\tilde{E}_{\mathbf{q}}$  to account for the iterative feedback effects. Following the previous study,<sup>19</sup> we have self-consistently determined these renormalized energies  $\tilde{\xi}_{\mathbf{k}}$ ,  $\tilde{E}_{\mathbf{q}}$  by solving numerically the flow equations (34) and (35) for the tight-binding lattice model  $\varepsilon_{\mathbf{k}} = -2t[\cos ak_x + \cos ak_y] - 2t_z \cos ck_z$  assuming reduced mobility along the  $z$  axis  $t_z = 0.1t$ . Initially (at  $l = 0$ ), we have assumed bosons to be localized. To establish some correspondence with the recent QMC studies,<sup>22</sup> we have used the same total concentration of carriers 0.16 and imposed the coupling  $g = 0.2D$  (where  $D = 8t$ ).

The important changeover of the boson dispersion  $\tilde{E}_{\mathbf{q}}$  upon varying temperature is shown in Fig. 2. We noticed that, below some characteristic temperature  $k_B T^* \sim 0.05D$ , there occurred a considerable reduction of the in-plane boson mass, defined as  $d^2 \tilde{E}_{\mathbf{q}} / dq_x^2 = \hbar^2 / m^B$ . Its temperature dependence [compared to the bare planar mass of fermions  $m_0^F = \hbar^2 / 2ta^2$ ] is illustrated in Fig. 3. The mentioned suppression of the boson mass below  $T^*$  coincided with appearance of the pseudogap in the fermion spectrum near  $\mu$ ; this property has been discussed at length in our previous studies<sup>39,40</sup> where we formulated the flow equation procedure for the present model (31). Below the other temperature  $k_B T_c \sim 0.026D$ , the Bose-Einstein condensate appeared in the system and simultaneously the parabolic dispersion evolved into the collective sound-wave mode  $\tilde{E}_{\mathbf{q}} \propto |\mathbf{q}|$  (see the bottom panel in Fig. 2).

Evolution of the effective boson and fermion spectra revealed a substantial influence on the superfluid fraction. In Fig. 4, we show the temperature dependence of such  $n_s(T)$ . Below the temperature  $T^*$  (for here chosen set of the model parameters  $T^* \sim 2T_c$ ), we observed a gradual buildup of the superfluid fraction. Passing below  $T_c$ , the superfluid fraction exhibited a further, stronger enhancement manifesting an onset of the long-range phase coherence caused by appearance of the Bose-Einstein condensate of pairs. At still lower temperatures, i.e., deep in the superconducting state  $T \ll T_c$ , we observed some flattening of the superfluid

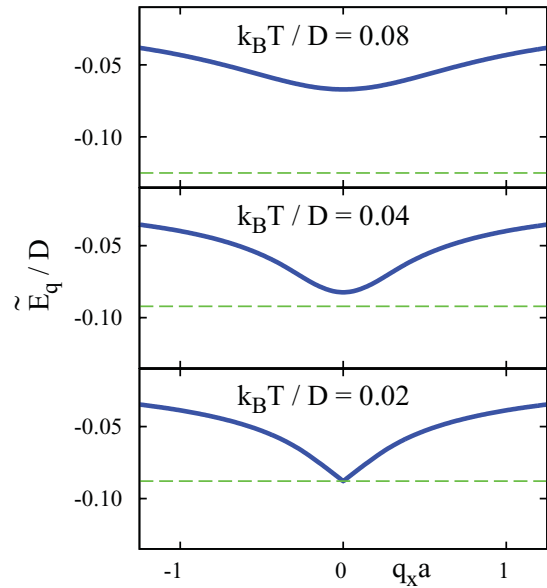


FIG. 2. (Color online) The renormalized boson energy  $\tilde{E}_{\mathbf{q}}$  obtained for three representative temperature regions:  $T > T^*$  (top panel),  $T^* > T > T_c$  (middle plot), and  $T_c > T$  (bottom panel). The dashed lines show the level  $2\mu(T)$ .

density rather than the expected linear dependence  $n_s(T \rightarrow 0) \simeq n_s(0) - \alpha T$  typical for  $d$ -wave superconducting systems with the Dirac-type excitations around the nodal points.<sup>44,45</sup> Presumably, these low-temperature results indicate that we are not correctly evaluating the transverse Fermi velocity  $v_{\Delta}$  and/or the longitudinal one  $v_f$ ; therefore, the proper linearity coefficient  $\alpha = [2 \ln(2)/\pi] v_f / v_{\Delta}$  (Ref. 44) is missing. We also suspect that, in our computations, we might apparently overestimate the role of antinodal areas, where the majority of bosons is effectively gathered for  $T \rightarrow 0$  (see Fig. 4 in Ref. 19). This artificial low-temperature dependence of  $n_s(T)$  needs a more careful investigation.

Summarizing this section, we have obtained the superfluid density  $n_s(T)$ , which clearly indicates a fairly broad temperature regime  $T_c < T < T^*$  of the Meissner rigidity appearing due to the superconducting fluctuations. Such fragile diamagnetism originates solely from the noncondensed preformed pairs, as has been previously suggested by several authors.<sup>46–48</sup>

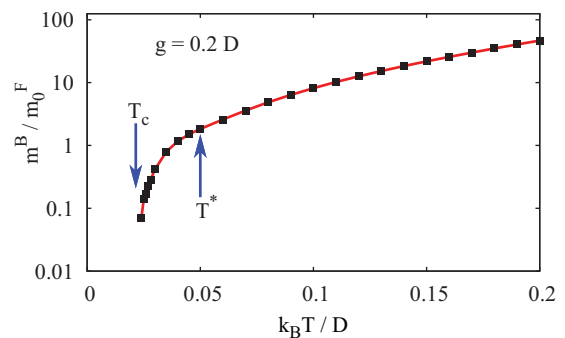


FIG. 3. (Color online) Temperature dependence of the effective boson mass  $m^B$  obtained for the initially discrete energy level  $E_{\mathbf{q}}(l = 0) = \text{const}$ . Our results resemble the QMC data shown in Fig. 9(a) of Ref. 22.

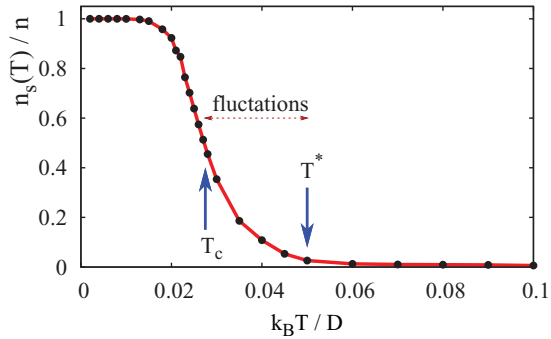


FIG. 4. (Color online) Temperature dependence of the superfluid fraction  $n_s(T)$  normalized to the total fermion concentration  $n$ . We can notice a broad regime where the fragile Meissner effect is caused by the superconducting fluctuations.

We have evaluated  $n_s(T)$  by means of the new nonperturbative method and determined the current-current response function (45) solving iteratively the set of flow equations. Our results seem to qualitatively capture the experimental data of the Ong group<sup>10</sup> and other recent measurements.<sup>49</sup>

## V. SUMMARY

We have addressed the linear response of the electron system with the pairing instabilities using nonperturbative framework of the continuous unitary transformation technique.<sup>1</sup> For the case of the Bose-Einstein condensed pairs, we have analytically derived the BCS result (30), which in the static and long-wavelength limit accounts for the Meissner effect. We have next extended such treatment onto the mixture of the noncondensed (preformed) pairs interacting with the mobile electrons through the charge-exchange Andreev scattering. We have determined the contributions (see Fig. 1) to the response function (45), where the vertices are expressed by the corresponding flow equations (38)–(41).

The central result (45) of our study generalizes the BCS current-current response function,<sup>26</sup> taking into account the residual diamagnetic effects originating from the finite-momentum preformed pairs. Such effects are studied here in an alternative way than the perturbative corrections due to the Aslamasov-Larkin and Maki-Thompson diagrams.<sup>13</sup> In our approach, the fluctuations enter the current-current response function through the convolution of one-boson and two-fermion propagators (see Fig. 1) instead of the higher-order convolutions typical for the standard diagrammatic study. In the present formalism, the influence of fluctuations affects the vertex functions, which have to be determined from the asymptotic solution of the flow equations (38)–(41).

In the static, long-wavelength limit, we find a clear evidence for the pronounced diamagnetic contribution, which might be relevant to the experimental data obtained for the underdoped cuprate materials in the lower part of the pseudogap state.<sup>10–12</sup> Our study is consistent with the recent quantum Monte Carlo results for the same model.<sup>22</sup> In both cases, the residual diamagnetism originates from the preformed pairs, the mobility of which considerably increases below  $T^*$ . Similar ideas concerning the noncondensed pairs have been emphasized by several other authors.<sup>23,46–48,50–52</sup>

In order to see a more specific relation of this treatment to the cuprate oxides, one should solve numerically the set of flow equations (38)–(41) for the realistic model with nearest-neighbor and next-nearest-neighbor hopping integrals. Another issue (not addressed here) concerns the doping effects that should affect the Fermi surface topology and influence the populations of the fermions and the preformed pairs.<sup>22,39</sup> It would also be worthwhile to solve the flow equations (38)–(41) fully self-consistently and investigate the electrodynamic properties using the response function  $K_{\alpha,\beta}(\mathbf{q},\omega)$ , which generalizes the standard BCS result.

We hope that the present formulation of the linear response theory by means of the flow equation method could stimulate further studies of the many-body effects in various subdisciplines of physics.

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## APPENDIX A: METHODOLOGY OF THE FLOW EQUATIONS

We give here a brief outline of the continuous canonical transformation for arbitrary Hamiltonian of the following structure:

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad (\text{A1})$$

where  $\hat{H}_0$  denotes the diagonal part (for instance, it can be the kinetic energy of particles) and  $\hat{H}_1$  stands for the off-diagonal term (i.e., interactions or any perturbations). Upon continuously transforming the Hamiltonian  $H(l) = U^\dagger(l) H U(l)$ , the  $l$  dependence (*flow*) is governed according to the following differential equation<sup>1</sup>:

$$\frac{d\hat{H}(l)}{dl} = [\hat{\eta}(l), \hat{H}(l)], \quad (\text{A2})$$

where the generating operator  $\hat{\eta}(l) \equiv \frac{d\hat{U}(l)}{dl} \hat{U}^{-1}(l)$ .

Choice of  $\eta(l)$  is usually dictated by the specific physical situation. One of the possibilities, suggested by Wegner,<sup>1</sup> is

$$\hat{\eta}(l) = [\hat{H}_0(l), \hat{H}_1(l)]. \quad (\text{A3})$$

It has been proved that (A3) guarantees

$$\lim_{l \rightarrow \infty} \hat{H}_1(l) = 0 \quad (\text{A4})$$

provided that no degeneracies are encountered. Several alternative proposals for  $\hat{\eta}(l)$  are capable to deal with the divergences,<sup>2</sup> the degenerate states,<sup>53</sup> or offer various other advantages that have been discussed in Ref. 3.

To carry out the statistical averages of the observables

$$\langle \hat{O} \rangle_{\hat{H}} = \text{Tr}\{e^{-\beta\hat{H}} \hat{O}\} / \text{Tr}\{e^{-\beta\hat{H}}\}, \quad (\text{A5})$$

(where  $\beta^{-1} = k_B T$ ) it is convenient to use the invariance of trace on the unitary transformations

$$\text{Tr}\{e^{-\beta\hat{H}} \hat{O}\} = \text{Tr}\{e^{-\beta\hat{H}(l)} \hat{O}(l)\}, \quad (\text{A6})$$



where  $\hat{O}(l) = \hat{U}(l)\hat{O}\hat{U}^{-1}(l)$ . The  $l$  dependence of  $\hat{O}(l)$  is imposed through the flow equation<sup>1</sup>

$$\frac{d\hat{O}(l)}{dl} = [\hat{\eta}(l), \hat{O}(l)] \quad (\text{A7})$$

similar to (A2) for  $\hat{H}(l)$ . Since the Hamiltonian  $\hat{H}(l)$  becomes diagonal for  $l \rightarrow \infty$ , the easiest way to compute the trace (A6) is with respect to  $\hat{H}(\infty)$ . This, however, requires that simultaneously with the continuous diagonalization of the Hamiltonian, one has to analyze the *flow* of other physical observables  $\hat{O} \rightarrow \hat{O}(l) \rightarrow \hat{O}(\infty)$ .

## APPENDIX B: EFFECTIVE QUASIPARTICLES ABOVE $T_c$

To determine the single-particle excitation spectrum for the model (31), we have to transform the individual operators  $\hat{c}_{\mathbf{k}\sigma}^{(\dagger)}(l) \equiv \hat{U}(l)\hat{c}_{\mathbf{k}\sigma}^{(\dagger)}\hat{U}^{-1}(l)$ , which is a bit tricky because  $\hat{U}(l)$  is not known explicitly. Following the scheme outlined in Sec. II B and using the operator  $\hat{\eta}(l)$  chosen in the form (32), we deduce the following ansatz for the fermion operators<sup>40</sup>:

$$\begin{aligned} \hat{c}_{\mathbf{k}\uparrow}(l) &= u_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}}(l) \hat{c}_{-\mathbf{k}\downarrow}^\dagger \\ &+ \frac{1}{\sqrt{N}} \sum_{\mathbf{q} \neq 0} [u_{\mathbf{k},\mathbf{q}}(l) \hat{b}_{\mathbf{q}}^\dagger \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow} + v_{\mathbf{k},\mathbf{q}}(l) \hat{b}_{\mathbf{q}} \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^\dagger], \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} \hat{c}_{-\mathbf{k}\downarrow}^\dagger(l) &= -v_{\mathbf{k}}^*(l) \hat{c}_{\mathbf{k}\uparrow} + u_{\mathbf{k}}^*(l) \hat{c}_{-\mathbf{k}\downarrow}^\dagger \\ &\times \frac{1}{\sqrt{N}} \sum_{\mathbf{q} \neq 0} [-v_{\mathbf{k},\mathbf{q}}^*(l) \hat{b}_{\mathbf{q}}^\dagger \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow} + u_{\mathbf{k},\mathbf{q}}^*(l) \hat{b}_{\mathbf{q}} \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^\dagger], \end{aligned} \quad (\text{B2})$$

where  $u_{\mathbf{k}}(0) = 1$  and the other coefficients are vanishing at  $l = 0$ . These  $l$ -dependent coefficients can be derived from Eq. (A7) for  $\hat{c}_{\mathbf{k}\uparrow}(l)$  and  $\hat{c}_{-\mathbf{k}\downarrow}^\dagger(l)$  operators.

The corresponding set of flow equations reads as<sup>40</sup>

$$\begin{aligned} \frac{du_{\mathbf{k}}(l)}{dl} &= \sqrt{n_{\mathbf{q}=0}^B} \alpha_{-\mathbf{k},\mathbf{k}}(l) v_{\mathbf{k}}(l) \\ &+ \frac{1}{N} \sum_{\mathbf{q} \neq 0} \alpha_{\mathbf{q}-\mathbf{k},\mathbf{k}}(l) (n_{\mathbf{q}}^B + n_{\mathbf{q}-\mathbf{k}\downarrow}^F) v_{\mathbf{k},\mathbf{q}}(l), \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} \frac{dv_{\mathbf{k}}(l)}{dl} &= -\sqrt{n_{\mathbf{q}=0}^B} \alpha_{\mathbf{k},\mathbf{k}}(l) u_{\mathbf{k}}(l) \\ &- \frac{1}{N} \sum_{\mathbf{q} \neq 0} \alpha_{\mathbf{k},\mathbf{q}+\mathbf{k}}(l) (n_{\mathbf{q}}^B + n_{\mathbf{q}+\mathbf{k}\uparrow}^F) u_{\mathbf{k},\mathbf{q}}(l), \end{aligned} \quad (\text{B4})$$

$$\frac{du_{\mathbf{k},\mathbf{q}}(l)}{dl} = \alpha_{\mathbf{q}-\mathbf{k},\mathbf{k}}(l) v_{\mathbf{k}}(l), \quad (\text{B5})$$

$$\frac{dv_{\mathbf{k},\mathbf{q}}(l)}{dl} = -\alpha_{\mathbf{k},\mathbf{q}+\mathbf{k}}(l) u_{\mathbf{k}}(l). \quad (\text{B6})$$

They are additionally coupled to the flow equations (33)–(35) because of the terms  $\alpha_{\mathbf{k},\mathbf{k}'}(l)$ . If one neglects the finite-momentum boson states [when  $u_{\mathbf{k},\mathbf{q}}(l) = 0 = v_{\mathbf{k},\mathbf{q}}(l)$ ], these equations can be solved analytically,<sup>40</sup> reproducing the standard BCS case discussed in Sec. II B. The case of  $\mathbf{q} \neq \mathbf{0}$  bosons is more cumbersome. We have previously studied such a problem numerically,<sup>39,40</sup> in particular, considering also the two-dimensional square lattice<sup>19</sup> with the tight-binding dispersion  $\xi_{\mathbf{k}}(0) = -2t[\cos(k_x a) + \cos(k_y a)] - 4t'\cos(k_x a)\cos(k_y a) - \mu$  assuming the initial discrete energy  $E_{\mathbf{q}}(0) = E_0$  and fixing the total charge concentration  $2 \sum_{\mathbf{q}} n_{\mathbf{q}}^B + \sum_{\mathbf{k}} (n_{\mathbf{k}\uparrow}^F + n_{\mathbf{k}\downarrow}^F)$ .

One of the valuable results obtained from such a formalism concerned the pseudogap regime. Since above  $T > T_c$  the condensate fraction is absent, we can notice that (B4) and (B5) imply  $v_{\mathbf{k}}(l) = 0 = u_{\mathbf{k},\mathbf{q}}(l)$ . In other words, the parametrizations (B1) and (B2) simplify then to

$$\hat{c}_{\mathbf{k}\uparrow}(l) = u_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow} + \frac{1}{\sqrt{N}} \sum_{\mathbf{q} \neq 0} v_{\mathbf{k},\mathbf{q}}(l) \hat{b}_{\mathbf{q}} \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^\dagger, \quad (\text{B7})$$

$$\hat{c}_{-\mathbf{k}\downarrow}^\dagger(l) = u_{\mathbf{k}}^*(l) \hat{c}_{-\mathbf{k}\downarrow}^\dagger - \frac{1}{\sqrt{N}} \sum_{\mathbf{q} \neq 0} v_{\mathbf{k},\mathbf{q}}^*(l) \hat{b}_{\mathbf{q}}^\dagger \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow}, \quad (\text{B8})$$

with the coefficients obeying

$$\begin{aligned} \frac{du_{\mathbf{k}}(l)}{dl} &= \frac{1}{N} \sum_{\mathbf{q} \neq 0} \alpha_{\mathbf{q}-\mathbf{k},\mathbf{k}}(l) (n_{\mathbf{q}}^B + n_{\mathbf{q}-\mathbf{k}\downarrow}^F) v_{\mathbf{k},\mathbf{q}}(l), \\ \frac{dv_{\mathbf{k},\mathbf{q}}(l)}{dl} &= -\alpha_{\mathbf{k},\mathbf{q}+\mathbf{k}}(l) u_{\mathbf{k}}(l). \end{aligned}$$

Under such circumstances, we thus find that the single-particle spectral function  $A(\mathbf{k}, \omega) = -\pi^{-1} \text{Im} G_{\sigma}(\mathbf{k}, \omega + i0^+)$  takes the following structure:

$$\begin{aligned} A(\mathbf{k}, \omega) &= |\tilde{u}_{\mathbf{k}}|^2 \delta(\omega - \tilde{\xi}_{\mathbf{k}}) \\ &+ \frac{1}{N} \sum_{\mathbf{q} \neq 0} (n_{\mathbf{q}}^B + n_{\mathbf{q}-\mathbf{k}\downarrow}^F) |\tilde{v}_{\mathbf{k},\mathbf{q}}|^2 \delta(\omega + \tilde{\xi}_{\mathbf{q}-\mathbf{k}} - \tilde{E}_{\mathbf{q}}). \end{aligned} \quad (\text{B9})$$

The first part of (B9) describes the quasiparticle whose renormalized dispersion  $\tilde{\xi}_{\mathbf{k}}$  in the temperature regime  $T < T_{sc}^*$  is discontinuous at the chemical potential (signaling a depletion of the low-energy states). The second part of Eq. (B9) contributes the shadow to the above-mentioned quasiparticle. These contributions constitute the characteristic Bogoliubov-type excitation spectrum, which has been indeed observed experimentally in the yttrium<sup>31</sup> and lanthanum<sup>32</sup> compounds.

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