

Flow equation approach to the pairing problems

T. Domański

Institute of Physics, M. Curie Skłodowska University, 20-031 Lublin, Poland

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We apply the flow equation method for studying the fermion systems where pairing interactions can either trigger the BCS instability with the symmetry breaking manifested by the off-diagonal order parameter or lead to the gaped single particle spectrum without any symmetry breaking. We construct the continuous Bogoliubov transformation in a scheme resembling the renormalization group procedure. We further extend this continuous transformation to a case where fermion pairs interact with the boson field. Due to temporal quantum fluctuations the single particle excitation spectrum develops a gap which is centered around the renormalized boson energy. When bosons undergo the Bose Einstein condensation this structure evolves into the BCS spectrum.

The flow equation method proposed a couple of years ago by Wegner [1] and independently by Wilson and Głazek [2] turned out to be a very useful theoretical tool in the studies of condensed matter and high energy physics (for a recent survey of applications see for instance [3]). The main idea of this technique is to transform the Hamiltonian to a diagonal or to a block-diagonal structure using the continuous unitary transformation with a flexibly adjusted generating operator.

A continuous diagonalization can be thought of as a process of renormalizing the coupling constants in a way similar to the Renormalization Group (RG) approach [4]. Difference between these methods is such that using the flow equation procedure one does not integrate out the high energy excitations (fast modes). Instead of this, they are renormalized in the initial part of the transformation. On the other hand the low energy excitations (slow modes) are transformed mainly at the very end. In the condensed matter these low energy excitations are most relevant for the physical properties.

In a first part of this paper we show how to construct the continuous unitary transformation for the reduced BCS model. Exact solution can be obtained of course by various ways, in particular via the Bogoliubov-Valatin transformation [5]. We rederive here the rigorous solution using the flow equation method. This worthwhile to do because the pairing instability problem is in the RG studies not a trivial issue. Only very recently there appeared in the literature several concepts based on the functional RG [6, 7] discussing how to work out the Cooper instability. Since the flow equation method is relative to the RG scheme we think it would be instructive to follow Wegner *et al* [8] and show how one can deal with the pairing interactions.

In the second part we generalize this transformation to a case of fermions interacting with the boson field. Such situation can refer for example to the conduction band electrons coupled through the Andreev type scattering to the localized electron pairs [9]. On more general grounds it can also describe some boson type modes interacting with the correlated electrons [10]. The other realization of this scenario is possible in the ultracold gases where fermion atoms interact with the weakly bound molecules giving rise to the Feshbach resonance and ultimately lead-

ing to the resonant superfluidity [11].

I. THE BCS PAIRING

In order to have a simple example how the fast and slow energy modes are scaled during the continuous unitary transformation we consider first the exactly solvable bilinear Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}, \sigma} \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger + h.c. \right) \quad (1)$$

describing fermions coupled to some pairing field $\Delta_{\mathbf{k}}$. In (1) we use the standard notation for the creation (annihilation) operator $\hat{c}_{\mathbf{k}\sigma}^\dagger$ ($\hat{c}_{\mathbf{k}\sigma}$) and energy $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ is measured from the chemical potential μ . Pairing field can be thought of, for instance, as a mean field approximation to the weak pairing potential between electrons $\Delta_{\mathbf{k}} = -\sum_{\mathbf{q}} V_{\mathbf{k}, \mathbf{q}} \langle \hat{c}_{-\mathbf{q}\downarrow} \hat{c}_{\mathbf{q}\uparrow} \rangle$, where $V_{\mathbf{k}, \mathbf{q}} < 0$. In general, one can treat $\Delta_{\mathbf{k}}$ as the boson operator e.g. arising from the Hubbard Stratonovich transformation for the interacting fermion system. It can also correspond to some boson field responsible for mediating the pairing between electrons. Coupling to the boson operator will be discussed in the next section.

Hamiltonians of the fermion and/or boson systems with a bilinear structure (1) have been already considered using the flow equation method [1, 8, 12, 13]. However, the authors have focused so far on determination of the quasiparticles energies. Here we supplement their analysis by calculating the correlation functions which contain information about the order parameter as well as a characteristics of the excitation spectrum.

It is well known, that the bilinear Hamiltonian (1) can be diagonalized by the (single step) Bogoliubov transformation [5]. All physical quantities, static and dynamical, can hence be determined exactly. In this work we get the same rigorous solution in a process of continuous diagonalization of the coupled states $|\mathbf{k}, \uparrow\rangle$ and $|\mathbf{k}, \downarrow\rangle$. How fast this can be achieved depends on a distance from the Fermi surface $|\mathbf{k} - \mathbf{k}_F|$ (or on the energy $\xi_{\mathbf{k}}$). Simultaneously with diagonalization there emerge the coherence factors $u_{\mathbf{k}}, v_{\mathbf{k}}$. Their final values depend on the relative distance of the momentum \mathbf{k} from the Fermi surface.

We construct the continuous unitary transformation $\hat{U}(l)$, where l stands for a formal *flow parameter* varying between the initial $l = 0$ value and value $l > 0$ such, that transformed Hamiltonian $\hat{H}(l) = \hat{U}(l)\hat{H}\hat{U}^{-1}(l)$ simplifies to the needed structure (diagonal, block-diagonal, tridiagonal or any other). With an increase of the flow parameter l the Hamiltonian evolves according to the general flow equation [1]

$$\frac{d\hat{H}(l)}{dl} = [\hat{\eta}(l), \hat{H}(l)] \quad (2)$$

where $\hat{\eta}(l) \equiv \frac{d\hat{U}(l)}{dl}\hat{U}^{-1}(l)$. We now require the transformed Hamiltonian $\hat{H}(l)$ to preserve its bilinear structure (1) and let only the coupling constants to become l -dependent (renormalized)

$$\begin{aligned} \hat{H}(l) = & \sum_{\mathbf{k}, \sigma} \xi_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger + h.c. \right) \\ & + \text{const}(l) \end{aligned} \quad (3)$$

with the initial conditions $\xi_{\mathbf{k}}(0) = \xi_{\mathbf{k}}$, $\Delta_{\mathbf{k}}(0) = \Delta_{\mathbf{k}}$ and $\text{const}(0) = 0$. This strategy reminds the scheme of RG approach. It can be shown [1] that, roughly speaking, the *flow parameter* l corresponds to Λ^{-1} .

There are several ways for choosing the generating operator $\hat{\eta}(l)$. In order to drive the Hamiltonian to a diagonal structure we adopt the Wegner's proposal [1] $\hat{\eta}(l) = [\hat{H}_0(l), \hat{H}(l)]$, where $\hat{H}_0(l) = \sum_{\mathbf{k}, \sigma} \xi_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma}$ denotes a diagonal part. With this choice the Hamiltonian becomes diagonal in the limit $l \rightarrow \infty$. In explicit form the generating operator for (3) reads

$$\hat{\eta}(l) = -2 \sum_{\mathbf{k}} \xi_{\mathbf{k}}(l) \left(\Delta_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger - h.c. \right). \quad (4)$$

Substituted into the equation (2) it gives

$$\begin{aligned} \frac{d\hat{H}(l)}{dl} = & 4 \sum_{\mathbf{k}, \sigma} \xi_{\mathbf{k}}(l) |\Delta_{\mathbf{k}}(l)|^2 \left(\hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} - 1 \right) \\ & - 4 \sum_{\mathbf{k}} (\xi_{\mathbf{k}}(l))^2 \left(\Delta_{\mathbf{k}}^*(l) \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger + h.c. \right) \end{aligned} \quad (5)$$

which is identical with the following set of flow equations

$$\frac{d\xi_{\mathbf{k}}(l)}{dl} = 4\xi_{\mathbf{k}}(l) |\Delta_{\mathbf{k}}(l)|^2 \quad (6)$$

$$\frac{d\Delta_{\mathbf{k}}(l)}{dl} = -4(\xi_{\mathbf{k}}(l))^2 \Delta_{\mathbf{k}}^*(l) \quad (7)$$

and $d \text{const}(l)/dl = -4 \sum_{\mathbf{k}, \sigma} \xi_{\mathbf{k}}(l) |\Delta_{\mathbf{k}}(l)|^2$.

Formally, solution of the equation (7) can be given as

$$|\Delta_{\mathbf{k}}(l)| = |\Delta_{\mathbf{k}}| e^{-4 \int_0^l [\xi_{\mathbf{k}}(l')]^2} \quad (8)$$

which yields that $\lim_{l \rightarrow \infty} \Delta_{\mathbf{k}}(l) = 0$ for all $\mathbf{k} \neq \mathbf{k}_F$. When we multiply equation (6) by $\xi_{\mathbf{k}}(l)$ and equation (7) by $\Delta_{\mathbf{k}}(l)^*$ then their sum gives the following invariance

$$\frac{d}{dl} \{ (\xi_{\mathbf{k}}(l))^2 + |\Delta_{\mathbf{k}}(l)|^2 \} = 0. \quad (9)$$

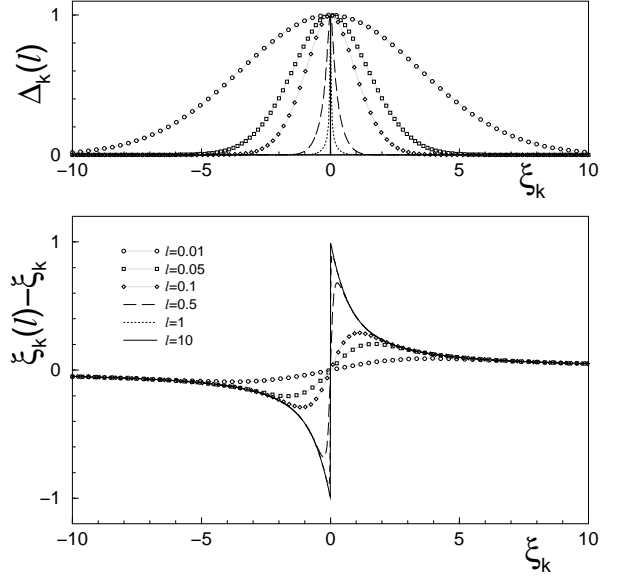


FIG. 1: Renormalization of $\Delta_{\mathbf{k}}(l)$ and $\xi_{\mathbf{k}}(l)$ during the *flow* for several values of l as denoted in the legend. At the starting point $l = 0$ we used the isotropic coupling $\Delta_{\mathbf{k}}(0) = \Delta$. Energies are expressed in units of Δ while l in units of Δ^{-2} .

This constraint together with vanishing of $\Delta_{\mathbf{k}}(l)$ in the limit $l \rightarrow \infty$ implies the known Bogoliubov spectrum

$$\xi_{\mathbf{k}}(\infty) = \text{sgn}(\xi_{\mathbf{k}}) \sqrt{(\xi_{\mathbf{k}})^2 + |\Delta_{\mathbf{k}}|^2}. \quad (10)$$

In figure 1 we plot the pairing field $\Delta_{\mathbf{k}}(l)$ and the change (renormalization) of fermion energies $\xi_{\mathbf{k}}(l) - \xi_{\mathbf{k}}$ at several stages of the continuous transformation. We notice, that fast modes (states distant from the Fermi energy) are transformed rather in the first part of the process and change of their energies is rather small. The slow modes (i.e. the low energy excitations) have to be worked on much longer. Asymptotically, at $l \rightarrow \infty$, the entire spectrum reduces to the Bogoliubov structure (10).

Now we turn attention to the dynamical quantities which can be expressed via the normal $\langle\langle \hat{c}_{\mathbf{k}\sigma}; \hat{c}_{\mathbf{k}\sigma}^\dagger \rangle\rangle_\omega$ and the anomalous single particle Green's function $\langle\langle \hat{c}_{\mathbf{k}\uparrow}; \hat{c}_{-\mathbf{k}\downarrow} \rangle\rangle_\omega$. We introduced the standard Fourier transforms for the retarded fermion Green's function $\int d\omega e^{i\omega t} \langle\langle \hat{A}; \hat{B} \rangle\rangle_\omega \equiv -i\Theta(t) \langle \hat{A}(t) \hat{B} + \hat{B} \hat{A}(t) \rangle$ and time evolution is given by $\hat{A}(t) = e^{it\hat{H}} \hat{A} e^{-it\hat{H}}$.

Thermal averaging for some arbitrary observable \hat{O} is defined as $\langle \hat{O} \rangle = \text{Tr} \{ e^{-\beta\hat{H}} \hat{O} \} / \text{Tr} \{ e^{-\beta\hat{H}} \}$, where $\beta^{-1} = k_B T$. Since the trace is invariant under unitary transformations we can write down

$$\begin{aligned} \text{Tr} \{ e^{-\beta\hat{H}} \hat{O} \} &= \text{Tr} \{ \hat{U}(l) e^{-\beta\hat{H}} \hat{O} \hat{U}^{-1}(l) \} \\ &= \text{Tr} \{ e^{-\beta\hat{H}(l)} \hat{O}(l) \}, \end{aligned} \quad (11)$$

where $\hat{O}(l) = \hat{U}(l) \hat{O} \hat{U}^{-1}(l)$. It is convenient to compute such trace in the limit $l = \infty$ because Hamiltonian becomes then diagonal. However, the price which

we pay for having the simplified Hamiltonian at $l = \infty$ is the additional necessity to transform the observables $\hat{O} \rightarrow \hat{O}(l) \rightarrow \hat{O}(\infty)$. This is to be done similarly as with the Hamiltonian (2) using

$$\frac{d\hat{O}(l)}{dl} = [\hat{\eta}(l), \hat{O}(l)]. \quad (12)$$

With the choice of the generating operator $\eta(l)$ given by (4) we find from the commutator $[\hat{\eta}(l), \hat{c}_{\mathbf{k}\uparrow}]$ that for $l > 0$ the annihilation operator $\hat{c}_{\mathbf{k}\uparrow}$ gets convoluted with $\hat{c}_{-\mathbf{k}\downarrow}^\dagger$. It is natural to propose the following Ansatz for the l -dependent operator

$$\hat{c}_{\mathbf{k}\uparrow}(l) = u_{\mathbf{k}}(l)\hat{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}}(l)\hat{c}_{-\mathbf{k}\downarrow}^\dagger. \quad (13)$$

Similar analysis for $\hat{c}_{-\mathbf{k}\downarrow}^\dagger$ operator leads to

$$\hat{c}_{-\mathbf{k}\downarrow}(l)^\dagger = -v_{\mathbf{k}}(l)\hat{c}_{\mathbf{k}\uparrow} + u_{\mathbf{k}}(l)\hat{c}_{-\mathbf{k}\downarrow}^\dagger \quad (14)$$

with the initial conditions $u_{\mathbf{k}}(0) = 1$, $v_{\mathbf{k}}(0) = 0$. These parametrized equations (13,14) substituted into (12) with $\eta(l)$ given in (4) lead to the coupled flow equations

$$\frac{du_{\mathbf{k}}(l)}{dl} = 2\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)v_{\mathbf{k}}(l), \quad (15)$$

$$\frac{dv_{\mathbf{k}}(l)}{dl} = -2\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)u_{\mathbf{k}}(l). \quad (16)$$

After straightforward algebra using equations (15,16) we obtain the following invariance $|u_{\mathbf{k}}(l)|^2 + |v_{\mathbf{k}}(l)|^2 = \text{const} = 1$ which assures that the l -dependent fermion operators (13,14) preserve the anticommutation relations

$$\{\hat{c}_{\mathbf{k}\sigma}(l), \hat{c}_{\mathbf{k}'\sigma'}^\dagger(l)\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}. \quad (17)$$

Without a loss of generality we assume $\Delta_{\mathbf{k}}(l)$ to be real so that the unknown coefficients $u_{\mathbf{k}}(l)$, $v_{\mathbf{k}}(l)$ become real too. To determine their asymptotic $l = \infty$ values we can rewrite (15) as

$$\frac{du_{\mathbf{k}}(l)}{v_{\mathbf{k}}(l)} = 2\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)dl \quad (18)$$

and we further integrate both sides in the limits $\int_{l=0}^{l=\infty}$. Using $v_{\mathbf{k}}(l) = \sqrt{1 - (u_{\mathbf{k}}(l))^2}$ we get for the l.h.s.

$$\int_{l=0}^{l=\infty} \frac{du_{\mathbf{k}}(l)}{\sqrt{1 - (u_{\mathbf{k}}(l))^2}} = \arccos u_{\mathbf{k}}(\infty) \quad (19)$$

due to $u_{\mathbf{k}}(0) = 1$. Using equation (7) we can replace $2\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)dl$ by $-d\Delta_{\mathbf{k}}(l)/2\xi_{\mathbf{k}}(l)$ and from the invariance (9) we obtain that $\xi_{\mathbf{k}}(l) = \sqrt{\xi_{\mathbf{k}}^2(\infty) - |\Delta_{\mathbf{k}}(l)|^2}$. The r.h.s. of equation (18) gives after integration

$$\begin{aligned} \int_{l=0}^{l=\infty} 2\xi_{\mathbf{k}}(l)\Delta_{\mathbf{k}}(l)dl &= - \int_{l=0}^{l=\infty} \frac{d\Delta_{\mathbf{k}}(l)}{2\sqrt{\xi_{\mathbf{k}}^2(\infty) - |\Delta_{\mathbf{k}}(l)|^2}} \\ &= \frac{-1}{2} \left(\arccos \frac{|\Delta_{\mathbf{k}}|}{|\xi_{\mathbf{k}}(\infty)|} - \frac{\pi}{2} \right) \end{aligned} \quad (20)$$

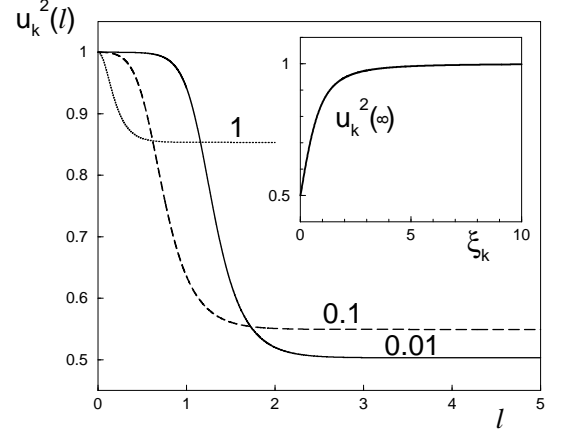


FIG. 2: Flow of the coherence factor $u_{\mathbf{k}}^2(l)$ for three fermion states of the initial energy $\xi_{\mathbf{k}} = 0.01, 0.1$ and 1 (in units Δ). Inset shows the effective value of the coherence factor $u_{\mathbf{k}}^2(\infty)$ versus $\xi_{\mathbf{k}}$.

because $\Delta_{\mathbf{k}}(\infty) = 0$. Combining the results (19,20)

$$2 \arccos u_{\mathbf{k}}(\infty) = \frac{\pi}{2} - \arccos \frac{|\Delta_{\mathbf{k}}|}{|\xi_{\mathbf{k}}(\infty)|} \quad (21)$$

we finally determine the $l = \infty$ factors

$$u_{\mathbf{k}}^2(\infty) = \frac{1}{2} \left[1 + \frac{\xi_{\mathbf{k}}}{\xi_{\mathbf{k}}(\infty)} \right] = 1 - v_{\mathbf{k}}^2(\infty). \quad (22)$$

One can simply show that $2u_{\mathbf{k}}(\infty)v_{\mathbf{k}}(\infty) = \Delta_{\mathbf{k}}/|\xi_{\mathbf{k}}(\infty)|$.

In figure 2 we present the coherence factor $u_{\mathbf{k}}^2(l)$ as a function of the flow parameter l . Again we notice that for momenta distant from the Fermi surface the coherence factors evolve rather quickly from the initial value $u_{\mathbf{k}}^2(0) = 1$ to the asymptotic values (see the inset). The coherence factors of the slow modes establish later on, similarly to renormalization of the corresponding excitation energies shown in figure 1. Saturation occurs around $l \propto 1/\xi_{\mathbf{k}}(\infty)^2$.

Since the transformed Hamiltonian $\hat{H}(\infty)$ is diagonal we can easily derive the single particle Green's functions (and the higher order Green's functions too). With the parameterizations (13,14) we obtain

$$\langle\langle \hat{c}_{\mathbf{k}\uparrow}; \hat{c}_{\mathbf{k}\uparrow}^\dagger \rangle\rangle_{\omega} = \frac{u_{\mathbf{k}}^2(\infty)}{\omega - \xi_{\mathbf{k}}(\infty)} + \frac{v_{\mathbf{k}}^2(\infty)}{\omega + \xi_{\mathbf{k}}(\infty)} \quad (23)$$

$$\langle\langle \hat{c}_{\mathbf{k}\uparrow}; \hat{c}_{-\mathbf{k}\downarrow} \rangle\rangle_{\omega} = \frac{u_{\mathbf{k}}(\infty)v_{\mathbf{k}}(\infty)}{\omega - \xi_{\mathbf{k}}(\infty)} - \frac{u_{\mathbf{k}}(\infty)v_{\mathbf{k}}(\infty)}{\omega + \xi_{\mathbf{k}}(\infty)}. \quad (24)$$

The equal time expectation values defined by $\langle\hat{B}\hat{A}\rangle = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f(\omega, T) \text{Imag}\langle\langle \hat{A}; \hat{B} \rangle\rangle_{\omega+i0^+}$ (where $f(\omega, T) = [e^{\beta\omega} + 1]^{-1}$ is the Fermi distribution function) yield the equations for average momentum occupancy

$$\langle\hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{k}\uparrow}\rangle = \frac{1}{2} \left[1 - \frac{\xi_{\mathbf{k}}}{\xi_{\mathbf{k}}(\infty)} \tanh \frac{\xi_{\mathbf{k}}(\infty)}{2k_B T} \right], \quad (25)$$

and for the off-diagonal order parameter

$$\langle \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} \rangle = \frac{\Delta_{\mathbf{k}}}{2\xi_{\mathbf{k}}(\infty)} \tanh \frac{\xi_{\mathbf{k}}(\infty)}{2k_B T}. \quad (26)$$

These equations (25,26) exactly reproduce the rigorous solution of the reduced BCS Hamiltonian.

In appendix A we explain how to generalize the present treatment to account for the scattering of finite momentum fermion pairs.

II. COUPLING TO THE BOSON MODE

We apply here the same method to the non-trivial problem describing itinerant fermions coupled to a zero dispersionless boson mode

$$\hat{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \hat{b} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger + h.c. \right) + \Omega \hat{b}^\dagger \hat{b}. \quad (27)$$

Here the boson mode can be regarded for instance as a pairing field derived from the Hubbard Stratonovich transformation in the system of interacting fermions. Model (27) can describe also the Andreev tunneling between c -fermions and some pair reservoir denoted by b -particles. Other possibility is to think of the case where itinerant fermions (e.g. conduction band electrons) coexist and interact with some localized fermion pairs [9]. The model (27) is also often applied to describe the ultracold fermion atoms coupled to the weakly bound molecules effectively leading to resonant Feshbach scattering [14].

To keep a conserved total number of particles $\hat{N}_{tot} = \sum_{\mathbf{k},\sigma} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} + 2\hat{b}^\dagger \hat{b}$ we apply the grand canonical ensemble. Ω stands for the boson energy measured from 2μ and $g_{\mathbf{k}}$ denotes the coupling constant. It can be shown that physical properties of this model depend only on a magnitude of $g_{\mathbf{k}}$, in other words all $|g_{\mathbf{k}}|e^{i\phi_{\mathbf{k}}}$ lead to identical results independently of phase $\phi_{\mathbf{k}}$.

In the mean field treatment of the Hamiltonian (27) one usually introduces the linearization $\hat{b} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger \simeq \langle \hat{b} \rangle \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger + \hat{b} (\hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger - \langle \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger \rangle)$ [9]. Such idea is based on assumption that there exists a finite amount of the Bose Einstein (BE) condensation of b particles. Hamiltonian (27) simplifies then to the ordinary BCS problem, where $\Delta \equiv -g \langle \hat{b} \rangle$. Let us however remark that the BE condensation of infinitely heavy boson (characterized by a discrete energy level Ω) cannot occur. Our present analysis based on the flow equation procedure shows a possible route to go beyond such mean field approximation.

We construct the continuous canonical transformation which decouples c from b particles. This can be achieved only approximately by choosing the following generating operator

$$\hat{\eta}(l) = \sum_{\mathbf{k}} [2\xi_{\mathbf{k}}(l) - \Omega(l)] g_{\mathbf{k}}(l) \hat{b} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger - h.c. \quad (28)$$

Applying (28) to the flow equation (2) and going through appropriate normal ordering we obtain the l -dependent Hamiltonian

$$\begin{aligned} \hat{H}(l) &\simeq \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(g_{\mathbf{k}}(l) \hat{b} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger + h.c. \right) \\ &+ \Omega(l) \hat{b}^\dagger \hat{b} + \sum_{\mathbf{k},\mathbf{k}'} U_{\mathbf{k},\mathbf{k}'}(l) \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{-\mathbf{k}\downarrow}^\dagger \hat{c}_{-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{k}'\uparrow} \end{aligned} \quad (29)$$

with $U_{\mathbf{k},\mathbf{k}'}(0) = 0$. In a straightforward way we derive the following set of flow equations

$$\frac{d\xi_{\mathbf{k}}(l)}{dl} = 2|g_{\mathbf{k}}(l)|^2 [2\xi_{\mathbf{k}}(l) - \Omega(l)] n^B \quad (30)$$

$$\frac{d\Omega(l)}{dl} = 2 \sum_{\mathbf{k}} |g_{\mathbf{k}}(l)|^2 [2\xi_{\mathbf{k}}(l) - \Omega(l)] (2n_{\mathbf{k}\sigma}^F - 1) \quad (31)$$

$$\frac{dg_{\mathbf{k}}(l)}{dl} = -g_{\mathbf{k}}(l) [2\xi_{\mathbf{k}}(l) - \Omega(l)]^2 \quad (32)$$

$$\begin{aligned} \frac{dU_{\mathbf{k},\mathbf{k}'}(l)}{dl} &= -g_{\mathbf{k}} g_{\mathbf{k}'}^*(l) [2\xi_{\mathbf{k}}(l) - \Omega(l)] \\ &- g_{\mathbf{k}'} g_{\mathbf{k}}^*(l) (2\xi_{\mathbf{k}'}(l) - \Omega(l)). \end{aligned} \quad (33)$$

Here $n_{\mathbf{k}\sigma}^F$ denotes the average fermion occupancy of the state $|\mathbf{k}, \sigma\rangle$ and $n^B = \langle \hat{b}^\dagger \hat{b} \rangle$. From formal solution of the equation (32) $g_{\mathbf{k}}(l) = g_{\mathbf{k}} \exp\{-\int_0^l dl' [2\xi_{\mathbf{k}}(l') - \Omega(l')]\}$ we notice that $g_{\mathbf{k}}(l \rightarrow \infty) = 0$.

In order to close the set of flow equations (30-33) we must determine the distribution function $n_{\mathbf{k}\sigma}^F$. From analysis of the flow equation (12) for the annihilation and creation fermion operators we come to a conclusion that

$$\hat{c}_{\mathbf{k}\uparrow}(l) = u_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}}(l) \hat{b} \hat{c}_{-\mathbf{k}\downarrow}^\dagger, \quad (34)$$

$$\hat{c}_{-\mathbf{k}\downarrow}^\dagger(l) = -v_{\mathbf{k}}(l) \hat{b}^\dagger \hat{c}_{\mathbf{k}\uparrow} + u_{\mathbf{k}}(l) \hat{c}_{-\mathbf{k}\downarrow}^\dagger. \quad (35)$$

Here the l -dependent factors satisfy the following flow equations

$$\frac{du_{\mathbf{k}}(l)}{dl} = v_{\mathbf{k}}(l) g_{\mathbf{k}}(l) [2\varepsilon_{\mathbf{k}}(l) - \Omega(l)] (n^B + n_{-\mathbf{k}\downarrow}^F) \quad (36)$$

$$\frac{dv_{\mathbf{k}}(l)}{dl} = -u_{\mathbf{k}}(l) g_{\mathbf{k}}(l) [2\varepsilon_{\mathbf{k}}(l) - \Omega(l)]. \quad (37)$$

with the boundary condition $u_{\mathbf{k}}(0) = 1$ and $v_{\mathbf{k}}(0) = 0$. For simplicity in (36,37) we neglected the terms proportional to $g_{\mathbf{k}}(l) U_{\mathbf{k}_1, \mathbf{k}_2}(l)$ which might eventually contribute some higher order corrections of the order $\sim g_{\mathbf{k}}^3$. Combining (36) and (37) we obtain the invariance

$$|u_{\mathbf{k}}(l)|^2 + (n^B + n_{-\mathbf{k}\downarrow}^F) |v_{\mathbf{k}}(l)|^2 = \text{const} = 1 \quad (38)$$

which guarantees that operators $\hat{c}_{\mathbf{k}\sigma}^{(\dagger)}(l)$ defined in (34,35) obey the anticommutation relations.

Various expectation values $\langle \hat{O} \rangle$ are easy to carry out in the limit $l \rightarrow \infty$ because fermions are decoupled from the boson field. Some delicate problem causes the two-body interaction $U_{\mathbf{k},\mathbf{k}'}(\infty)$. As will be shown below this potential is small, so in the lowest order perturbation theory we

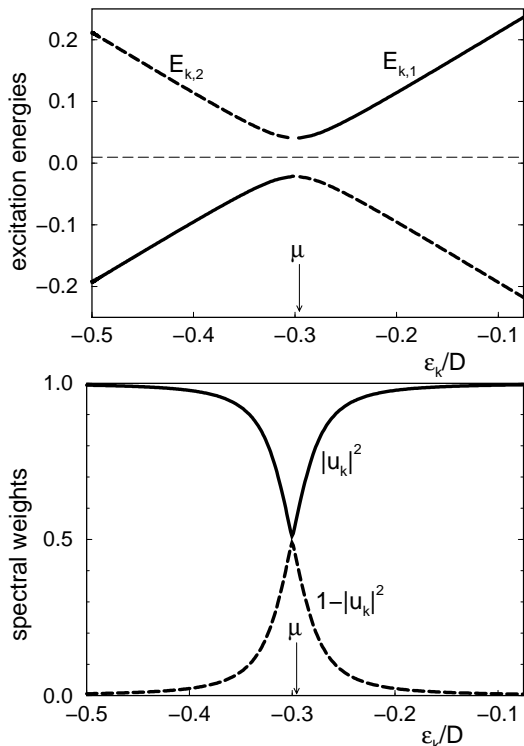


FIG. 3: The single particle excitation energies (top panel) and their corresponding spectral weights (bottom panel). The thin dashed line indicates that the excitations $E_{k,\nu}$ are gaped around the renormalized boson energy $\frac{1}{2}\Omega(\infty)$. We used for computations $g_{\mathbf{k}} = 0.05D$, $\Omega/2 + \mu = -0.3D$, $k_B T = 0.015D$.

incorporate its effect via the Hartree shift $U_{\mathbf{k},\mathbf{k}}(\infty)n_{\mathbf{k}\sigma}^F$ to fermion energies. This weak point of the present analysis can be however systematically improved.

Using the parameterization (34,35) we find the normal single particle Green's function

$$\langle\langle \hat{c}_{\mathbf{k}\uparrow}; \hat{c}_{\mathbf{k}\uparrow}^\dagger \rangle\rangle_\omega = \frac{|u_{\mathbf{k}}(\infty)|^2}{\omega - E_{\mathbf{k},1}} + (n^B + n_{-\mathbf{k}\downarrow}^F) \frac{|v_{\mathbf{k}}(\infty)|^2}{\omega - E_{\mathbf{k},2}} \quad (39)$$

with quasiparticle energies $E_{\mathbf{k},1} = \xi_{\mathbf{k}}(\infty) + U_{\mathbf{k},\mathbf{k}}(\infty)n_{\mathbf{k}\sigma}^F$ and $E_{\mathbf{k},2} = \Omega(\infty) - E_{\mathbf{k},1}$. The corresponding spectral function $A(\mathbf{k}, \omega) = -\frac{1}{\pi} \text{Im} \langle\langle \hat{c}_{\mathbf{k}\uparrow}; \hat{c}_{\mathbf{k}\uparrow}^\dagger \rangle\rangle_{\omega+i0}$ satisfies the sum rule $\int_{-\infty}^{\infty} d\omega A(\mathbf{k}, \omega) = 1$ due to invariance (38). Finally, the momentum distribution function is thus given by

$$n_{\mathbf{k}\sigma}^F = \frac{|u_{\mathbf{k}}(\infty)|^2}{e^{\beta E_{\mathbf{k},1}} + 1} + (n^B + n_{-\mathbf{k}\downarrow}^F) \frac{|v_{\mathbf{k}}(\infty)|^2}{e^{\beta E_{\mathbf{k},2}} + 1}. \quad (40)$$

We solved numerically the set of coupled flow equations (30-33,36,37) and (40) using the Runge Kutta algorithm. In our calculations we considered the finite fermion band $-D/2 \leq \epsilon_{\mathbf{k}} \leq D/2$ and assumed the flat density of states. Total particle concentration was fixed for $\langle \hat{N}_{tot} \rangle = 1$ which corresponds to nearly equal populations of fermions and bosons $\sum_{\mathbf{k}} n_{\mathbf{k}\sigma}^F \sim n^B$.

In the top panel of figure (3) we show the single particle fermion spectrum which turns out to be gaped. Two

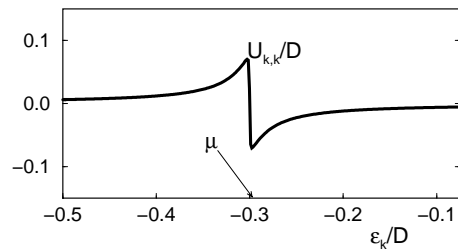


FIG. 4: Potential $U_{\mathbf{k},\mathbf{k}}(\infty)$ of the two-body interaction between fermions for the same set of parameters as in figure 3.

branches of the excitations $E_{\mathbf{k},\nu}$ are discontinuous around the energy $\Omega(\infty)/2$ (the dashed line). For here considered situation it is located slightly above the chemical potential μ and is dependent on temperature. We checked that the value of the gap is rather independent of T . The corresponding spectral weights are shown in the bottom panel of figure 3. They behave similar to the BCS formfactors but we notice only a negligible dependence on temperature. Figure 4 additionally confirms that the two-body potential is indeed weak.

We would like to emphasize, that the gaped spectrum shown in figure 3 is not related to any off-diagonal order parameter. According to the parameterization (34, 35) we easily determine that the anomalous Green's function $\langle\langle \hat{c}_{-\mathbf{k}\downarrow}; c_{\mathbf{k}\uparrow}^\dagger \rangle\rangle$ is identically zero. The above mentioned structure should hence be referred as a pseudogap. The broken symmetry can arise if and only if a certain fraction of b particles becomes BE condensed [16]. There are two possibilities for this to occur: a) either we assume bosons to be mobile right from the outset of the problem, or b) we allow for a finite momentum exchange $\mathbf{q} \neq \mathbf{0}$ in the interaction term $\hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^\dagger \hat{b}_{\mathbf{q}}$.

The second option has been explored in the literature by several groups using various many-body techniques (for representative list of the references see the review paper [17]). The purpose of our present study was to show that the temporal quantum fluctuations alone can induce the (pseudo)gap structure - there is no need for involving spatial fluctuations. Of course in real systems there always exist both, temporal and spatial fluctuations. The latter cause that upon lowering temperature the (pseudo)gap smoothly evolves into the true gap of superconducting state [18].

III. CONCLUSIONS

By means of the flow equation method [1] we analyzed the bilinear Hamiltonians describing fermion systems with pairing interactions. This new technique becomes more and more popular [3] owing to its conceptual simplicity which allows to go beyond the frame of various perturbative approximations [19]. This method is based on the continuous canonical transformations in course of which all the parameters are gradually renormalized. In

some way this can be compared to renormalization of the coupling constants within the RG procedure [4]. The high energy excitations are here renormalized mainly in the beginning, while the low energy excitations must be worked on until the end of transformation.

First part of our paper illustrates how to reproduce the rigorous solution of the BCS model using a continuous version of the Bogoliubov transformation [5]. In particular, we show how one can approach the symmetry broken problems which usually is not an easy task for the RG methods [6, 7]. This scheme can be extended for analysis of the two-body interactions where in principle various kinds of instabilities can arise. Some results for the 2-dimensional Hubbard model have been already reported [20] with use of the flow equation method.

In the second part we focused on the case, where fermion pairs are coupled to some single level boson field. Such situation can take place when the correlated fermion system is affected by bosonic modes such as e.g. the pairing fluctuations in the systems of reduced dimensionality (for instance the HTS cuprates) [21]. Moreover, this sort of physics is recently intensively studied for the ultracold fermion atoms where interaction with the weakly bound molecules leads to the Feshbach resonance which is a driving mechanism for the atomic superfluidity [22].

We solved selfconsistently the corresponding set of flow equations with a help of the numerics. We showed that interactions are responsible for appearance of a gap in the single particle fermion spectrum. This gap is centered around the boson energy $\frac{1}{2}\Omega(\infty)$ instead of the chemical potential. It does not signify any symmetry breaking because no order parameter is present in the system. The off-diagonal order parameter can eventually appear if bosons have a finite mobility (characterized by some dispersion $\Omega_{\mathbf{q}}$ different from the discrete energy Ω considered here) and if those bosons undergo the BE condensation [16]. The center of gap moves then to the chemical potential as has been explained in the previous study [15]. Here, we emphasize that the single particle gap can appear in a normal state purely due to the temporal quantum fluctuations. In realistic systems with additional spatial fluctuations such normal state (pseudo)gap is expected to evolve smoothly into the gap of symmetry broken superconducting state.

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APPENDIX A

The continuous Bogoliubov transformation (discussed in section 1) can be extended to the more general bilinear

Hamiltonian

$$\hat{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} - \sum_{\mathbf{k},\mathbf{q}} \left(G_{\mathbf{k},\mathbf{q}} \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^\dagger + h.c. \right) \quad (\text{A1})$$

which reduces to (1) if $G_{\mathbf{k},\mathbf{q}} = \delta_{\mathbf{q},\mathbf{0}} \Delta_{\mathbf{k}}$. Besides $\mathbf{q}=\mathbf{0}$ this model allows also for a scattering of the finite momentum \mathbf{q} fermion pairs. In what follows below we briefly analyze the effect of such finite momentum scattering.

The generator from Wegner's proposal

$$\hat{\eta}^0(l) = - \sum_{\mathbf{k},\mathbf{q}} [\xi_{\mathbf{k}}(l) + \xi_{\mathbf{q}-\mathbf{k}}(l)] \left(G_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^\dagger \right) - h.c. \quad (\text{A2})$$

applied to the flow equation (2) induces the off-diagonal terms $c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{p}\neq\mathbf{k}\sigma}$. In order to eliminate them from the transformed Hamiltonian $H(l)$ we replace (A2) by

$$\hat{\eta}(l) = \hat{\eta}^0(l) + \sum_{\mathbf{k},\mathbf{q},\sigma} \gamma_{\mathbf{k},\mathbf{q},\sigma}(l) \left(\hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{q}\sigma} - \hat{c}_{\mathbf{q}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} \right). \quad (\text{A3})$$

After simple algebraic calculations we find that the off-diagonal terms are exactly canceled if

$$\gamma_{\mathbf{k},\mathbf{q},\uparrow}(l) = \sum_{\mathbf{k}'} \frac{\xi_{\mathbf{k}}(l) + \xi_{\mathbf{k}'}(l)}{\xi_{\mathbf{k}}(l) - \xi_{\mathbf{q}}(l)} G_{\mathbf{k}',\mathbf{k}'+\mathbf{q}-\mathbf{k}}(l) G_{\mathbf{k},\mathbf{q}}(l), \quad (\text{A4})$$

$$\gamma_{\mathbf{k},\mathbf{q},\downarrow}(l) = \sum_{\mathbf{k}'} \frac{\xi_{\mathbf{k}}(l) + \xi_{\mathbf{k}'}(l)}{\xi_{\mathbf{k}}(l) - \xi_{\mathbf{q}}(l)} G_{\mathbf{k}',\mathbf{k}+\mathbf{k}'}(l) G_{\mathbf{k}',\mathbf{k}'+\mathbf{q}}(l). \quad (\text{A5})$$

The modified generating operator (A3) has a virtue to preserve the initial structure of the Hamiltonian $\hat{H}(l) = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}}(l) \hat{c}_{\mathbf{k}\sigma}^\dagger \hat{c}_{\mathbf{k}\sigma} - \sum_{\mathbf{k},\mathbf{q}} \left(G_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{k}\uparrow}^\dagger \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^\dagger + h.c. \right) + const(l)$. The corresponding set of flow equations is

$$\frac{d\xi_{\mathbf{k}}(l)}{dl} = 2 \sum_{\mathbf{q}} (\xi_{\mathbf{k}}(l) + \xi_{\mathbf{q}-\mathbf{k}}(l)) |G_{\mathbf{k},\mathbf{q}}(l)|^2, \quad (\text{A6})$$

$$\frac{dG_{\mathbf{k},\mathbf{q}}(l)}{dl} = - (\xi_{\mathbf{k}}(l) + \xi_{\mathbf{q}-\mathbf{k}}(l))^2 G_{\mathbf{k},\mathbf{q}}^*(l) \quad (\text{A7})$$

$$+ 2 \sum_{\mathbf{k}',\sigma} (\gamma_{\mathbf{k},\mathbf{k}',\sigma}(l) - \gamma_{\mathbf{k}',\mathbf{k},\sigma}(l)) G_{\mathbf{k}',\mathbf{q}-\mathbf{k}+\mathbf{k}'}(l),$$

$$\frac{d const(l)}{dl} = -2 \sum_{\mathbf{k},\mathbf{q},\sigma} (\xi_{\mathbf{k}}(l) + \xi_{\mathbf{q}-\mathbf{k}}(l)) |G_{\mathbf{k},\mathbf{q}}(l)|^2. \quad (\text{A8})$$

We were not able to solve analytically the equations (A6-A8) therefore we explored them numerically assuming the tight binding dispersion $\xi_{\mathbf{k}} = -2t \cos(k_x a)$ and using the pairing potential $G_{\mathbf{k},\mathbf{q}}$ in a Lorentian form

$$G_{\mathbf{k},\mathbf{q}} = \mathcal{N}(n) \frac{\Delta}{n [q_x a]^2 + 1}. \quad (\text{A9})$$

The normalization factor $\mathcal{N}(n)$ was taken such that $\sum_{\mathbf{q}} G_{\mathbf{k},\mathbf{q}} = \Delta$ and we set $\Delta/4t = 0.01$.

Flow of the operators $\hat{c}_{\mathbf{k}\sigma}^{(\dagger)}(l)$ takes here the form

$$\hat{c}_{\mathbf{k}\uparrow}(l) = \sum_{\mathbf{q}} u_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow} + \sum_{\mathbf{q}} v_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow} \quad (\text{A10})$$

$$\hat{c}_{-\mathbf{k}\downarrow}^{\dagger}(l) = -\sum_{\mathbf{q}} v_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{q}+\mathbf{k}\uparrow} + \sum_{\mathbf{q}} u_{\mathbf{k},\mathbf{q}}(l) \hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}^{\dagger} \quad (\text{A11})$$

which generalizes the previous equations (13, 14) due to finite momentum scattering. Substituting (A10,A11) into the flow equation (12) we obtain

$$\begin{aligned} \frac{du_{\mathbf{k},\mathbf{q}}(l)}{dl} &= [\xi_{\mathbf{k}}(l) + \xi_{\mathbf{q}-\mathbf{k}}(l)] G_{\mathbf{k},\mathbf{q}}(l) v_{\mathbf{k},\mathbf{0}}(l) \\ &+ \sum_{\mathbf{p} \neq \mathbf{k}} [\xi_{\mathbf{k}-\mathbf{q}}(l) + \xi_{\mathbf{q}+\mathbf{p}}(l)] G_{\mathbf{q}-\mathbf{k},2\mathbf{q}-\mathbf{k}+\mathbf{p}}(l) v_{\mathbf{k},\mathbf{q}}(l), \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} \frac{dv_{\mathbf{k},\mathbf{q}}(l)}{dl} &= -[\xi_{\mathbf{k}}(l) + \xi_{\mathbf{q}-\mathbf{k}}(l)] G_{\mathbf{k},\mathbf{q}}(l) u_{\mathbf{k},\mathbf{0}}(l) \\ &- \sum_{\mathbf{p} \neq \mathbf{k}} [\xi_{\mathbf{k}-\mathbf{q}}(l) + \xi_{\mathbf{q}+\mathbf{p}}(l)] G_{\mathbf{q}-\mathbf{k},2\mathbf{q}-\mathbf{k}+\mathbf{p}}(l) u_{\mathbf{k},\mathbf{q}}(l). \end{aligned} \quad (\text{A13})$$

From these equations (A12-A13) we derive the invariance

$$\sum_{\mathbf{q}} |u_{\mathbf{k},\mathbf{q}}(l)|^2 + \sum_{\mathbf{q}} |v_{\mathbf{k},\mathbf{q}}(l)|^2 = 1 \quad (\text{A14})$$

which again leads to the anticommutation relations (17).

With use of the parameterization (A10,A11) we can now determine the single particle Green's functions. Let us first consider the diagonal part (in the Nambu representation). The spectral function $A(\mathbf{k},\omega) = -\pi^{-1} \text{Imag} \langle \langle \hat{c}_{\mathbf{k}\uparrow}; \hat{c}_{\mathbf{k}\uparrow}^{\dagger} \rangle \rangle_{\omega+i0^+}$ turns out to consist of two contributions

$$A(\mathbf{k},\omega) = A_{coh}(\mathbf{k},\omega) + A_{inc}(\mathbf{k},\omega). \quad (\text{A15})$$

The coherent part

$$\begin{aligned} A_{coh}(\mathbf{k},\omega) &= |u_{\mathbf{k},\mathbf{0}}(\infty)|^2 \delta[\omega - \xi_{\mathbf{k}}(\infty)] \\ &+ |v_{\mathbf{k},\mathbf{0}}(\infty)|^2 \delta[\omega + \xi_{-\mathbf{k}}(\infty)] \end{aligned} \quad (\text{A16})$$

describes the long-lived quasiparticle modes of energies $\pm \xi_{\mathbf{k}}(\infty)$ similar to the Bogoliubov modes (10) discussed previously for $G_{\mathbf{k},\mathbf{q}} = \Delta_{\mathbf{k}} \delta_{\mathbf{q},\mathbf{0}}$. The remaining incoherent part

$$\begin{aligned} A_{inc}(\mathbf{k},\omega) &= \sum_{\mathbf{q} \neq \mathbf{0}} |u_{\mathbf{k},\mathbf{q}}(\infty)|^2 \delta[\omega - \xi_{\mathbf{q}+\mathbf{k}}(\infty)] \\ &+ \sum_{\mathbf{q} \neq \mathbf{0}} |v_{\mathbf{k},\mathbf{q}}(\infty)|^2 \delta[\omega + \xi_{\mathbf{q}-\mathbf{k}}(\infty)] \end{aligned} \quad (\text{A17})$$

corresponds to the background spectrum which is spread over the large energy regime. Due to invariance (A14)

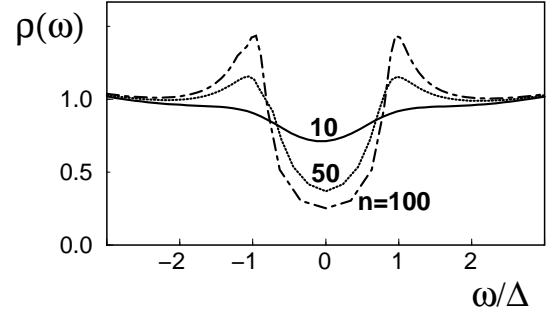


FIG. 5: The local density of states $\rho(\omega) = \sum_{\mathbf{k}} A(\mathbf{k},\omega)$ of fermions coupled to the potential $G_{\mathbf{k},\mathbf{q}}$ given in (A9) with $n = 10, 50$ and 100 . The density is normalized to $\rho_0(\omega)$ of the noninteracting system.

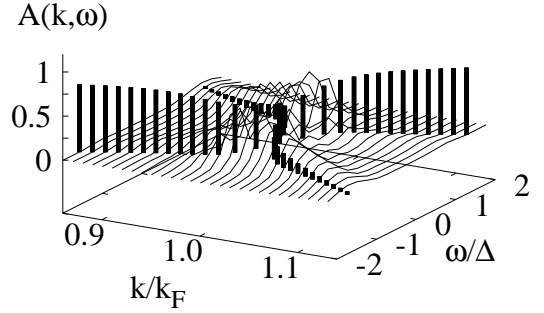


FIG. 6: The single particle spectral function $A(\mathbf{k},\omega)$ obtained for the potential (A9) with $n = 100$. The thick vertical lines represent the spectral weights $|u_{\mathbf{k},\mathbf{0}}(\infty)|^2$ and $|v_{\mathbf{k},\mathbf{0}}(\infty)|^2$ of the long-lived quasiparticles $\pm \xi_{\mathbf{k}}(\infty)$. The thin solid lines show ω dependence of the incoherent background $A_{inc}(\mathbf{k},\omega)$.

the total spectral weight of the coherent and incoherent parts satisfies the sum rule $\int_{-\infty}^{\infty} d\omega A(\mathbf{k},\omega) = 1$.

Figure 5 presents the local density of states calculated by us with use of the definition $\rho(\omega) = \sum_{\mathbf{k}} A(\mathbf{k},\omega)$. We notice that the excitation spectrum is not longer truly gaped. There occurs only a partial suppression of the fermion states around the energy ($\omega = 0$). This property should be assigned to the scattering of finite momentum fermion pairs on the potential $G_{\mathbf{k},\mathbf{q}}$ introduced ad hoc in (A1). Physically this means that only a certain fraction of the fermion states is expelled from a vicinity of the Fermi surface.

Figure (6) shows both parts of the spectral function for the pairing potential (A9) with $n = 100$, which yields the strongest suppression of the low lying fermion states plotted in the figure 5. Two modes of the coherent part have similar spectral weights as the BCS coherence factors. However, in distinction from the standard BCS solution, the corresponding quasiparticle dispersion $\pm \xi_{\mathbf{k}}(\infty)$ is in this case gapless. This is evidently caused

by a coupling to the finite momentum fermion pairs. As regards the incoherent background it builds up mainly near the Fermi surface \mathbf{k}_F . Such damped fermion states are located around the quasiparticle energies $\pm\xi_{\mathbf{k}}(\infty)$. We estimated that for $\mathbf{k} = \mathbf{k}_F$ the incoherent states contribute nearly 30 percent of the total spectral weight.

The off-diagonal single particle Green's function $\langle\langle\hat{c}_{\mathbf{k}\uparrow};\hat{c}_{-\mathbf{p}\downarrow}\rangle\rangle_{\omega}$ has a similar structure to (A17). Skipping the unnecessary technical details we show the expression for expectation value of the order parameter

$$\langle\hat{c}_{-\mathbf{p}\downarrow}\hat{c}_{\mathbf{k}\uparrow}\rangle = -\frac{1}{2}\sum_{\mathbf{q}}\left\{\frac{u_{\mathbf{k},\mathbf{q}+\mathbf{p}-\mathbf{k}}(\infty)v_{\mathbf{p},\mathbf{q}}(\infty)}{\exp[\beta\xi_{\mathbf{q}+\mathbf{p}}(\infty)]+1}-\frac{u_{\mathbf{p},\mathbf{q}}(\infty)v_{\mathbf{k},\mathbf{q}-\mathbf{p}+\mathbf{k}}(\infty)}{\exp[-\beta\xi_{\mathbf{q}-\mathbf{p}}(\infty)]+1}\right\}. \quad (\text{A18})$$

We investigated numerically the \mathbf{q} -dependence of the order parameter $\langle\hat{c}_{\mathbf{q}-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow}\rangle$ and we noticed its rather fast decrease against $|\mathbf{q}|$. For $n = 10, 50$ and 100 we obtained that magnitude of the order parameter at $|\mathbf{q}| = 0.2\pi/a$ is circa 100 times smaller than $\langle\hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow}\rangle$. This indicates that finite momentum fermion pairs are less favored in the system. Moreover, the finite momentum fermion pairs are damped entities because they arise from the incoherent part of the off-diagonal spectral function existing in a broad energy interval ω for momenta $\sim \mathbf{k}_F$.

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