1 Structure of White Dwarfs and Neutron Stars

1.1 Introduction

In 1967 Jocelyn Bell, a graduate student, along with her thesis advisor, Anthony Hewish, discovered the first pulsar, something from outer space that emits very regular pulses of radio energy. After recognizing that these pulse trains were so unvarying that they would not support an origin from LGM's (Little Green Men), it soon became generally accepted that the pulsar was due to radio emission from a rapidly rotating neutron star [1] endowed with a very strong magnetic field. By now more than a thousand pulsars have been catalogued [2]. Pulsars are by themselves quite interesting [3], but perhaps more so is the structure of the underlying neutron star. This lecture discusses a project for calculating structure which can be solved by a student writing either a Fortran or a Methematica program for solving the Tolman-Oppenheimer-Volkov (TOV) equations [4] to calculate masses and radii of neutron stars.

There is much more physics in the problem than just simply integrating a pair of coupled non-linear differential equations. In addition to the physics (and even some astronomy), the student must *think* about the sizes of things he or she is calculating, that is, believing and understanding the answers one gets. Another side benefit is that the student learns about the stability of numerical solutions and how to deal with singularities. In the process he or she also learns the inner mechanics of the calculational package (e.g., Mathematica) being used.

The student should begin with a derivation of the (Newtonian) coupled equations, and, presumably, become acquainted with the general relativistic (GR) corrections. Before trying to solve these equations, one needs to work out the relation between the energy density and pressure of the matter that constitutes the stellar interior, i.e., an equation of state (EoS). The first EoS's to try can be derived from the non-interacting Fermi gas model, which brings in quantum statistics (the Pauli exclusion principal) and special relativity. It is necessary to keep careful track of dimensions, and converting to dimensionless quantities is helpful in working these EoS's out.

As a warm-up problem the student can, at this point, integrate the Newtonian equations and learn about white dwarf stars. Putting in the GR corrections, one can then proceed in the same way to work out the structure of *pure* neutron stars (i.e., reproducing the results of Oppenheimer and Volkov [4]). It is interesting at this point to compare and see how important the GR corrections are, i.e., how different a neutron star is from that which would be given by classical Newtonian mechanics.

Realistic neutron stars, of course, also contain some protons and electrons. As a first approximation one can treat this multi-component system within the non-interacting Fermi gas model. In the process one learns about chemical potentials. To improve upon this treatment we must include nuclear interactions in addition to the degeneracy pressure from the Pauli exclusion principle that is used in the Fermi gas model. The nucleon-nucleon interaction is not familiar to a student in general, but there is a simple model for the nuclear matter EoS. It has parameters which are fit to quantities such as the binding energy per nucleon in symmetric nuclear matter, the so-called nuclear symmetry energy (it is really an asymmetry) and the (not so well known) nuclear compressibility. Working this out is also an excellent exercise, which even touches on the speed of sound (in nuclear matter). With these nuclear interactions in addition to the Fermi gas energy in the EoS, one finds (pure) neutron star masses and radii which are quite different from those using the Fermi gas EoS.

The above three paragraphs provide the outline of what follows in this lecture.

We should point out that there is a similar discussion of this matter by Balian and Blaizot [7]. Much of the material we discuss here is covered in the textbook by Shapiro and Teukolsky [8]. However, the emphasis here is on students learning through computation. One of our intentions is to establish here a framework for the student to interact with *his or her own* computer program, and in the process learn about the physical scales involved in the structure of compact degenerate stars.

2 The Tolman-Oppenheimer-Volkov Equation

2.1 Newtonian Formulation

A nice first exercise is to derive the following structure equations from classical Newtonian mechanics,

$$\frac{dp}{dr} = -\frac{G\rho(r)\mathcal{M}(r)}{r^2} = -\frac{G\epsilon(r)\mathcal{M}(r)}{c^2r^2} \tag{1}$$

$$\frac{d\mathcal{M}}{dr} = 4\pi r^2\rho(r) = \frac{4\pi r^2\epsilon(r)}{c^2} \tag{2}$$

$$\mathcal{M}(r) = 4\pi\int_0^r r'^2dr'\rho(r') = 4\pi\int_0^r r'^2dr'\epsilon(r')/c^2\,. \tag{3}$$

Here $G = 6.673\times10^{-8}$ dyne-cm^2/g^2 is Newton's gravitational constant, $\rho(r)$ is the mass density at the distance r (in gm/cm^3), and ϵ is the corresponding energy density (in ergs/cm^3) [9]. The quantity $\mathcal{M}(r)$ is the total mass inside the sphere of radius r. A sufficient hint for the derivation is shown in Fig. 1. (Challenge question: the above equations actually hold for any value of r, not just the large-r situation depicted in the figure. Can the student also do the derivation in spherical coordinates where the box becomes a cut-off wedge?)

Note that, in the second halves of these equations, we have departed slightly from Newtonian physics, defining the energy density $\epsilon(r)$ in terms of the mass density $\rho(r)$ according to the (almost) famous Einstein equation from special relativity,

$$\epsilon(r) = \rho(r)c^2\,. \tag{4}$$

This allows Eq. (1) to be used when one takes into account contributions of the interaction energy between the particles making up the star.

In what follows, we may inadvertently set $c = 1$ so that ρ and ϵ become indistinguishable. We'll try not to do that here so students following the equations in this presentation can keeping checking dimensions as they proceed. However, they might as well get used to this often-used physicists' trick of setting $c = 1$ (as well as $\hbar = 1$).

To solve this set of equations for $p(r)$ and $\mathcal{M}(r)$ one can integrate outwards from the origin ($r = 0$) to the point $r = R$ where the pressure

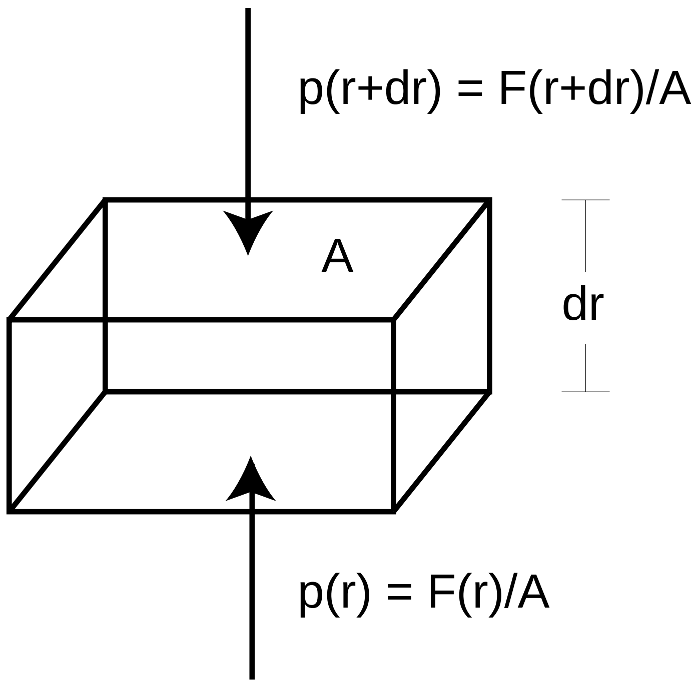

Figure 1: Diagram for derivation of Eq. (1)

goes to zero. This point defines R as the radius of the star. One needs an initial value of the pressure at $r = 0$, call it p_0, to do this, and R and the total mass of the star, $\mathcal{M}(R) \equiv M$, will depend on the value of p_0.

Of course, to be able to perform the integration, one also needs to know the energy density $\epsilon(r)$ in terms of the pressure $p(r)$. This relationship is the equation of state (EoS) for the matter making up the star. Thus, a lot of the student's effort in this project will necessarily be directed to developing an appropriate EoS for the problem at hand.

2.2 General Relativistic Corrections

The Newtonian formulation presented above works well in regimes where the mass of the star is not so large that it significantly "warps" space-time. That is, integrating Eqs. (1) and (2) will work well in cases when general relativistic (GR) effects are not important, such as for the compact stars known as white dwarfs. By creating a quantity using G that has dimensions of length, the student can see when it becomes important to include GR effects. (This happens when GM/c^2R becomes non-negligible.) As the student will see, this is the case for typical neutron stars.

It is probably not to be expected that an undergraduate physics major derive the GR corrections to the above equations. For that, one can look at various textbook derivations of the Tolman-Oppenheimer-Volkov (TOV) equation [5] [8]. It should suffice to simply state the corrections to Eq. (1) in terms of three additional (dimensionless) factors,

$$\frac{dp}{dr} = -\frac{G\epsilon(r)\mathcal{M}(r)}{c^2r^2}\left[1+\frac{p(r)}{\epsilon(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{\mathcal{M}(r)c^2}\right]\left[1-\frac{2G\mathcal{M}(r)}{c^2 r}\right]^{-1} . \quad (5)$$

The differential equation for $\mathcal{M}(r)$ remains unchanged. The first two factors in square brackets represent special relativity corrections of order v^2/c^2. This can be seen in that pressure p goes, in the non-relativistic limit, like $k_F^2/2m = mv^2/2$ (see Eq. (14) below) while ϵ and $\mathcal{M}c^2$ go like mc^2. That is, these factors reduce to 1 in the non-relativistic limit. (The student should have, by now, realized that p and ϵ have the same dimensions.) The last factor is a GR correction and the size of $G\mathcal{M}/c^2r$, as we emphasized above, determines whether it is important (or not).

Note that the correction factors are all positive definite. It is as if Newtonian gravity becomes stronger at every r. That is, special and general relativity strengthens the relentless pull of gravity!

The coupled non-linear equations for $p(r)$ and $\mathcal{M}(r)$ can also in this case be integrated from $r = 0$ for a starting value of p_0 to the point where $p(R) = 0$, to determine the star mass $M = \mathcal{M}(R)$ and radius R for this value of p_0. These equations invoke a balance between gravitational forces and the internal pressure. The pressure is a function of the EoS, and for certain conditions it may not be sufficient to withstand the gravitational attraction. Thus the structure equations imply there is a maximum mass that a star can have.

2.3 White Dwarf Stars

2.3.1 A Few Facts

Let us violate (in words only) the second law of thermodynamics by warming up on cold compact stars called white dwarfs. For these stellar objects, it suffices to solve the Newtonian structure equations, Eqs. (1)-(3) [10].

White dwarf stars [11] were first observed in 1844 by Friedrich Bessel (the same person who invented the special functions bearing that name). He noticed that the bright star Sirius wobbled back and forth and then deduced that the visible star was being orbited by some unseen object, i.e., it is a binary system. The object itself was resolved optically some 20 years later and thus *earned* the name of "white dwarf." Since then numerous other white (and the smaller brown) dwarf stars have been observed (or detected).

A white dwarf star is a low- or medium-mass star near the end of its lifetime, having burned up, through nuclear processes, most of its hydrogen and helium forming carbon, silicon or (perhaps) iron. They typically have a mass less than 1.4 times that of our Sun, $M_\odot = 1.989 \times 10^{33}$ g [12]. They are also much smaller than our Sun, with radii of the order of 10^4 km (to be compared with $R_\odot = 6.96 \times 10^5$ km). These values can be worked out from the period of the wobble for the dwarf-normal star binary in the usual Keplerian way. As a result (and as is also the case for neutron stars), the natural dimensions for discussing white dwarfs are for masses to be in units of solar mass, $M_\odot$, and distances to be in km. Using these numbers the student should be able to make a quick estimate of the (average) densities of our Sun and of a white dwarf, to get a feel for the numbers that he will be encountering.

Since $GM/c^2R \approx 10^{-4}$ for such a typical white dwarf, we can concentrate here on solving the non-relativistic structure equations of Sec. 2.1. (Question: why is it a good approximation to drop the special relativistic corrections for these dwarfs?)

The reason a dwarf star is small is because, having burned up all the nuclear fuel it can, there is no longer enough thermal pressure to prevent its gravity from crushing it down. As the density increases, the electrons in the atoms are pushed closer together, which then tend to fall into the lowest energy levels available to them. (The star begins to get colder.) Eventually the Pauli principle takes over, and the electron degeneracy pressure (to be discussed next) provides the means for stabilizing the star against its gravitational attraction [12, 8]. This is the physics behind the EoS which one needs to integrate the Newtonian structure equations above, Eqs. (1) and (2).

2.3.2 Fermi Gas Model for Electrons

For free electrons the number of states dn available at momentum k per unit volume is

$$dn = \frac{d^3k}{(2\pi\hbar)^3} = \frac{4\pi k^2 dk}{(2\pi\hbar)^3} \, . \tag{6}$$

(This is a result from their modern physics course that students should review if they don't remember it.) Integrating, one gets the electron number density

$$n = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^2 dk = \frac{k_F^3}{3\pi^2\hbar^3} \, . \tag{7}$$

The additional factor of two comes in because there are two spin states for each electron energy level. Here $k_F c$, the Fermi energy, is the maximum energy electrons can have in the star under consideration. It is a parameter which varies according to the star's total mass and its history, but which the student is free to set in the calculations he or she is about to make.

Each electron is neutralized by a proton, which in turn is accompanied in its atomic nucleus by a neutron (or perhaps a few more, as in the case of a nucleus like $^{56}\mathrm{Fe}_{26}$). Thus, neglecting the electron mass m_e with respect to the nucleon mass m_N, the mass density of the star is essentially given by

$$\rho = n m_N A/Z \, , \tag{8}$$

where A/Z is the number of nucleons per electron. For $^{12}\mathrm{C}$, $A/Z = 2.00$, while for $^{56}\mathrm{Fe}$, $A/Z = 2.15$. Note that, since n is a function of k_F, so is ρ. Conversely, given a value of ρ,

$$k_F = \hbar \left(\frac{3\pi^2 \rho}{m_N} \frac{Z}{A} \right)^{1/3} \, . \tag{9}$$

The energy density of this star is also dominated by the nucleon masses, i.e., $\epsilon \approx \rho c^2$.

The contribution to the energy density from the electrons (including their rest masses) is

$$\begin{aligned} \epsilon_{\mathrm{elec}}(k_F) &= \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} (k^2c^2 + m_e^2 c^4)^{1/2} k^2 dk \\ &= \epsilon_0 \int_0^{k_F/m_e c} (u^2+1)^{1/2} u^2 du \\ &= \frac{\epsilon_0}{8} \left[(2x^3 + x)(1+x^2)^{1/2} - \sinh^{-1}(x) \right] \, , \end{aligned} \tag{10}$$

where

$$\epsilon_0 = \frac{m_e^4 c^5}{\pi^2 \hbar^3} \tag{11}$$

carries the desired dimensions of energy per volume and $x = k_F/m_e c$. The *total* energy density is then

$$\epsilon = n m_N A/Z + \epsilon_{\rm elec}(k_F)\,. \tag{12}$$

One should check that the first term here is much larger than the second.

To get our desired EoS, we need an expression for the pressure. The following presents a problem (!) that the student should work through. From the first law of thermodynamics, $dU = dQ - pdV$, then at temperature T fixed at $T = 0$ (where $dQ = 0$ since $dT = 0$)

$$p = -\left.\frac{\partial U}{\partial V}\right]_{T=0} = n^2 \frac{d(\epsilon/n)}{dn} = n\frac{d\epsilon}{dn} - \epsilon = n\mu - \epsilon\,, \tag{13}$$

where the energy density here is the total one given by Eq. (12). The quantity $\mu_i = d\epsilon/dn_i$ defined in the last equality is known as the chemical potential of the electrons. This is a concept which will be especially useful in Section 5 where we consider an equilibrium mix of neutrons, protons and electrons.

Utilizing Eq. (10), Eq. (13) yields the pressure (another problem!)

$$\begin{aligned} p(k_F) &= \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_F}(k^2c^2 + m_e^2c^4)^{-1/2}k^4 dk \\ &= \frac{\epsilon_0}{3}\int_0^{k_F/m_e c}(u^2+1)^{-1/2}u^4 du \\ &= \frac{\epsilon_0}{24}\left[(2x^3 - 3x)(1+x^2)^{1/2} + 3\sinh^{-1}(x)\right]\,. \end{aligned} \tag{14}$$

(Hint: use the $n^2 d(\epsilon/n)/dn$ form and remember to integrate by parts.)

Using Mathematica [13] the student can show that the constant $\epsilon_0 = 1.42\times 10^{24}$ in units that, at this point, are erg/cm^3. (Yet another problem: verify that the units of ϵ_0 are as claimed [14].) One also finds that Mathematica can perform the integrals analytically. (We quoted the results already in the equations above.) They are a bit messy, however, as they both involve an inverse hyperbolic sine function, and thus are not terribly enlightening. It is useful, however, for the student to make a plot of ϵ versus p (such as shown in Fig. 2) for values of the parameter $0 \le k_F \le 2m_e$. This curve has a shape much like $\epsilon^{4/3}$ (the student should compare with this), and there is a good reason for that.

Consider the (relativistic) case when $k_F \gg m_e$. Then Eq. (14) simplifies to

$$\begin{aligned} p(k_F) &= \frac{\epsilon_0}{3}\int_0^{k_F/m_e c} u^3 du = \frac{\epsilon_0}{12}(k_F/m_e c)^4 = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z\rho}{m_N A}\right)^{4/3} \\ &\approx K_{\rm rel}\,\epsilon^{4/3}\,, \end{aligned} \tag{15}$$

where

$$K_{\rm rel} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{4/3}\,. \tag{16}$$

A star having simple EoS like $p = K\epsilon^\gamma$ is called a "polytrope", and we therefore see that the relativistic electron Fermi gas gives a polytropic EoS with $\gamma = 4/3$. As will be seen in the next subsection, a polytropic EoS allows one to solve the structure equations (numerically) in a relatively straight-forward way [15].

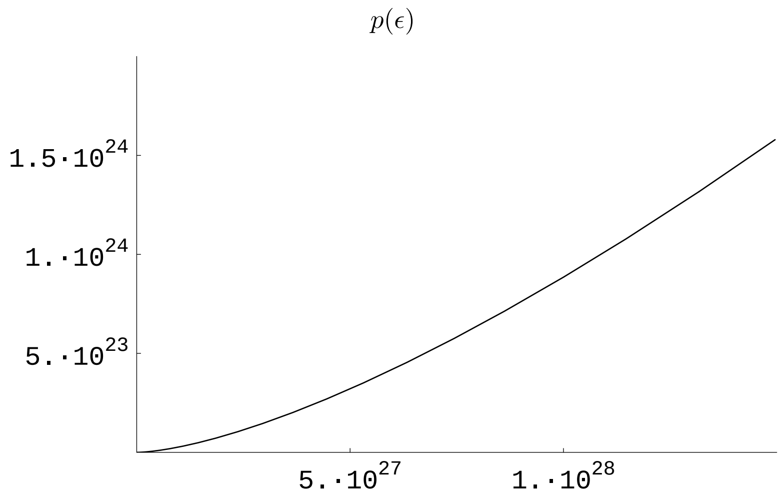

Figure 2: Relation between pressure p (y-axis) and energy density ϵ (x-axis) in the free electron Fermi gas model. Units are ergs/cm^3. Note that the pressure is much smaller than the energy density, since the latter is dominated by the massive nucleons.

There is another polytropic EoS for the non-interacting electron Fermi gas model corresponding to the non-relativistic limit, where $k_F \ll m_e$. In a way similar to the derivation of Eq. (15), one finds

$$p = K_{\mathrm{non-rel}} \epsilon^{5/3}\,, \quad \text{where} \quad K_{\mathrm{non-rel}} = \frac{\hbar^2}{15\pi^2 m_e} \left(\frac{3\pi^2 Z}{A m_N c^2} \right)^{5/3} . \tag{17}$$

[Question: what are the units of K_{rel} and $K_{\mathrm{non-rel}}$? Task: confirm that in the appropriate limits, Eqs. (10) and (14) reduce to those in Eqs. (15) and (17).]

2.3.3 The Structure Equations for a Polytrope

As mentioned earlier, we want to express our results in units of km and $\mathrm{M}_\odot$. Thus it is useful to define $\bar{\mathcal{M}}(r) = \mathcal{M}(r)/\mathrm{M}_\odot$. The first Newtonian structure equation, Eq. (1), then becomes

$$\frac{dp(r)}{dr} = -R_0 \frac{\epsilon(r) \bar{\mathcal{M}}(r)}{r^2}\,, \tag{18}$$

where the constant $R_0 = G\mathrm{M}_\odot/c^2 = 1.47$ km. (That is, for those who already know, R_0 is one half the Schwartzschild radius of our sun.) In this equation p and ϵ still carry dimensions of, say, ergs/cm^3. Therefore, let us define dimensionless energy density and pressure, $\bar{\epsilon}$ and $\bar{p}$, by

$$p = \epsilon_0 \bar{p}\,, \quad \epsilon = \epsilon_0 \bar{\epsilon} \tag{19}$$

where ϵ_0 has dimensions of energy density. This ϵ_0 is not the same as defined in Eq. (11). Its numerical choice here is arbitrary, and a suitable strategy is to make that choice based on the dimensionful numbers that define the problem at hand. We'll employ this strategy to fix it below. For a polytrope, we can write

$$\bar{p} = \bar{K}\bar{\epsilon}^{\gamma}\,, \quad \text{where} \;\; \bar{K} = K{\epsilon_0}^{\gamma-1} \;\; \text{is dimensionless.} \tag{20}$$

It is easier to solve Eq. (18) for $\bar{p}$, so we should express $\bar{\epsilon}$ in terms of it,

$$\bar{\epsilon} = (\bar{p}/\bar{K})^{1/\gamma}\,. \tag{21}$$

Equation (18) can now be recast in the form

$$\frac{d\bar{p}(r)}{dr} = -\frac{\alpha\,\bar{p}(r)^{1/\gamma}\bar{\mathcal{M}}(r)}{r^2}\,, \tag{22}$$

where the constant α is

$$\alpha = R_0/\bar{K}^{1/\gamma} = R_0/(K\epsilon_0^{\gamma-1})^{1/\gamma}\,. \tag{23}$$

Equation (22) has dimensions of 1/km, with α in km (since R_0 is). That is, it is to be integrated with respect to r, with r also in km.

We can choose any convenient value for α since ϵ_0 is still free. For a given value of α, ϵ_0 is then fixed at

$$\epsilon_0 = \left[\frac{1}{K}\left(\frac{R_0}{\alpha}\right)^{\gamma}\right]^{1-\gamma}\,. \tag{24}$$

We also need to cast the other coupled equation, Eq. (2), in terms of dimensionless $\bar{p}$ and $\bar{\mathcal{M}}$,

$$\frac{d\bar{\mathcal{M}}(r)}{dr} = \beta r^2 \bar{p}(r)^{1/\gamma}\,, \tag{25}$$

where [16]

$$\beta = \frac{4\pi\epsilon_0}{\mathrm{M}_{\odot}c^2\bar{K}^{1/\gamma}} = \frac{4\pi\epsilon_0}{\mathrm{M}_{\odot}c^2(K\epsilon_0^{\gamma-1})^{1/\gamma}}\,. \tag{26}$$

Equation (25) also carries dimensions of 1/km, the constant β having dimesnions 1/km^3. Note that, in integrating out from $r = 0$, the initial value of $\bar{\mathcal{M}}(0) = 0$.

2.3.4 Integrating the Polytrope Numerically

Our task is to integrate the coupled first-order differential equations (DE), Eqs. (22) and (25), out from the origin, $r = 0$, to the point R where the pressure falls to zero, $\bar{p}(R) = 0$ [17]. To do this we need two initial values, $\bar{p}(0)$ (which must be positive) and $\bar{\mathcal{M}}(0)$ (which we already know must be 0). The star's radius, R, and its mass $M = \bar{\mathcal{M}}(R)$ in units of $\mathrm{M}_{\odot}$ will vary, depending on the choice for $\bar{p}(0)$.

For purposes of numerical stability in solving Eqs. (22) and (25), we want the constants α and β to be not much different from each other (and probably not much different from 1). We will see below that this can be arranged for both of the two polytropic EoS's discussed above for white dwarfs.

Our coupled DE's are quite non-linear. In fact, because of the $\bar{p}^{1/\gamma}$ factors, the solution will become complex when $\bar{p}(r) < 0$, i.e., when $r > R$. Thus we will want to recognize when this happens. How can this be programmed?

Mathematica and similar symbolic/numerical packages have built-in first-order DE solvers. Perhaps the solver is as simple as a fixed, equal-step Runge-Kutta routine (as in MathCad 7 Standard), but there are often more sophisticated solvers in more recent versions. These packages also allow for program control constructs such as do-loops, whiles and the like.

Thus, consider a do-loop on a variable $\bar{r}$ running in appropriately small steps over a range that is sure to contain the expected value of R. Call the DE solver inside this loop, integrating the coupled DE's from $r = 0$ to $\bar{r}$. When the solver routine exits, check to see if the last value of $\bar{p}$, i.e., $\bar{p}(\bar{r})$, has a real part which has gone negative. If so, then break out of the loop, calling $R = \bar{r}$. If not, go on to the next larger value of $\bar{r}$ and call the DE solver again.

More discussion of how to program the integration of the DE's is inappropriate here, since we want to encourage the student to learn from programming to appreciate how the symbolic/numerical package is used.

2.3.5 The Relativistic Case $k_F \gg m_e$

This is the regime for white dwarfs with the largest mass. A larger mass needs a greater central pressure to support it. However, large central pressures mean the squeezed electrons become relativistic.

Recall that the polytrope exponent $\gamma = 4/3$ for this case and the equation of state is given by $P = K_{\rm rel}\epsilon^{\gamma}$ with $K_{\rm rel}$ given by Eq. (16). After some trial and error, we chose in our program (the student may want to try something else)

$$\alpha = R_0 = 1.473 \text{ km} \qquad [k_F \gg m_e]\,, \tag{27}$$

which in turn fixes, from Eq. (24),

$$\epsilon_0 = 7.463 \times 10^{39} \text{ergs/cm}^3 = 4.17 \text{ M}_\odot c^2/\text{km}^3 \qquad [k_F \gg m_e]\,. \tag{28}$$

The first question the student should ask, in checking this number, is whether such a large number is physically reasonable.

Continuing with the $k_F \gg m_e$ numerics, Eqs. (16) and (26) give

$$\beta = 52.46\,/\text{km}^3 \qquad [k_F \gg m_e]\,, \tag{29}$$

which is about 30 times larger than α, but probably manageable from the standpoint of performing the numerical integration.

In *our* first attempt to integrate the coupled DE's for this case (using a do-loop as described above) we chose $\bar{p}(0) = 1.0$. This gives us a white dwarf of radius $R \approx 2$ km, which is miniscule compared with the expected radius of $\approx 10^4$ km! Why? What went wrong?

The student who also makes this kind of mistake will eventually realize that our choice of scale, $\epsilon_0 = 4.17 \text{M}_\odot c^2/\text{km}^3$, represents a *huge* energy density. One can simply estimate the average energy density of a star with a 10^4 km radius and a mass of one solar mass by the ratio of its rest mass energy to its volume,

$$\langle\epsilon\rangle \approx \frac{\text{M}_\odot c^2}{R^3} = 10^{-12} \text{ M}_\odot c^2/\text{km}^3\,, \tag{30}$$

which is much, much smaller than the ϵ_0 here. In addition, the pressure p is about 2000 times smaller than the energy density ϵ (see Fig. 2). Thus, choosing a starting value of $\bar{p}(0) \sim 10^{-15}$ would probably be more physical.

Table 1: Radius R (in km) and mass M (in $M_\odot$) for white dwarfs with a relativistic electron Fermi gas EoS.

$\bar{p}(0)$	R	M
10^{-14}	4840	1.2431
10^{-15}	8600	1.2432
10^{-16}	15080	1.2430

Doing so does give much more reasonable results. Table 1 shows our program's results for R and M and how they depend on $\bar{p}(0)$. The surprise here is that, within the numerical error expected, all these cases have the *same* mass! Increasing the central pressure doesn't allow the star to be more massive, just more compact.

It turns out that this result is correct, i.e., that the white dwarf mass is independent of the choice of the central pressure. It is not easy to see this, however, from the numerical integration we have done here. The discussion in terms of Lane-Emden functions [15] shows why, though the mathematics here might be a bit steep for many undergraduates. For this reason, we quote without proof the analytic results. For the case of a polytropic equation of state $p = K\epsilon^\gamma$, the mass

$$M = 4\pi\epsilon^{2(\gamma-\frac{4}{3})/3} \left(\frac{K\gamma}{4\pi G(\gamma-1)}\right)^{3/2} \zeta_1^2 \, |\theta(\zeta_1)| \,, \tag{31}$$

and the radius

$$R = \sqrt{\frac{K\gamma}{4\pi G(\gamma-1)}} \, \zeta_1 \, \epsilon^{(\gamma-2)/2} \,. \tag{32}$$

In the above-mentioned solutions, ζ_1 and $\theta(\zeta_1)$ are numerical constants that depend on the polytropic index γ. By examining Eq. (31), we see that for $\gamma = 4/3$ the mass is independent of the central energy density, and hence also the central pressure p_0. Also, note that from Eq. (32), the radius decreases with increasing central pressure as $R \propto p_0^{(\gamma-2)/2\gamma} = p_0^{-1/4}$. In any case, the student should notice this point and use it as check of the numerical results obtained. Figure 3 shows the dependence of $\bar{p}(r)$ and $\bar{\mathcal{M}}(r)$ on distance for the case $\bar{p}(0) = 10^{-16}$. It is interesting that $\bar{p}(r)$ becomes small and essentially flat around 8000 km before finally going through zero at $R = 15,080$ km.

The results and graphs shown here were generated with a Mathematica 4.0 program, but we were able to reproduce them using MathCad 7 Standard. In that case, however, programming a loop is difficult, so we searched by hand for the endpoint (where the real part of $\bar{p}(r)$ goes negative). More recent versions of MathCad have more complete program constructs, such as while-loops, so this process could undoubtedly be automated. (Alternatively, the student might try to solve for a root of $\bar{p}(r) = 0$.)

2.3.6 The Non-Relativistic Case, $k_F \ll m_e$

Eventually, as the central pressure $\bar{p}(0)$ gets smaller, the electron gas is no longer relativistic. Also as the pressure gets smaller, it can support less mass. This moves us in the direction of the less massive white dwarfs, and, as it turns out, these dwarfs are *larger* (in radius) than the ones in the last section.

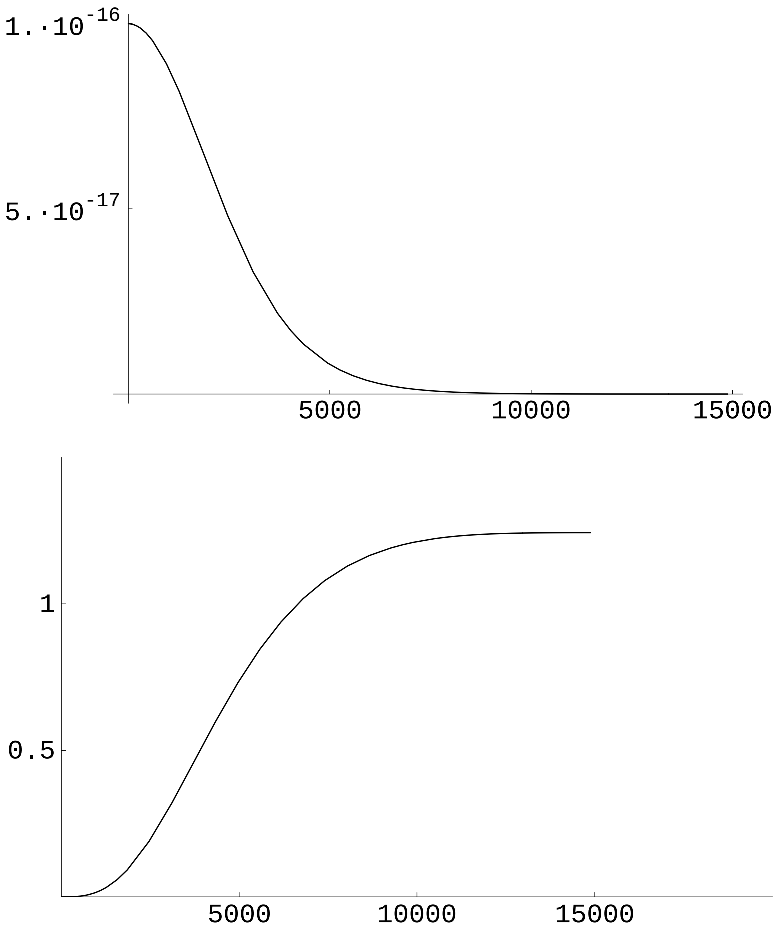

Figure 3: $\bar{p}(r)$ and $\bar{\mathcal{M}}(r)$ for white dwarfs using the relativistic electron Fermi gas model. Here $\bar{p}(0) = 10^{-16}$.

Table 2: Radius R (in km) and mass M (in $M_\odot$) for white dwarfs with a non-relativistic electron Fermi gas EoS.

$\bar{p}(0)$	R	M
10^{-15}	10620	0.3941
10^{-16}	13360	0.1974

In the extreme case, when $k_F \ll m_e$, we can integrate the structure equations for the other polytropic EoS, where $\gamma = 5/3$. The programming for this is very much the same as in the 4/3 case, but the numbers involved are quite different (as are the results).

Inserting the values of the physical constants in Eq. (17), we find

$$K_{\rm non-rel} = 3.309 \times 10^{-23}\ \frac{\rm cm^2}{\rm ergs^{2/3}}\,. \tag{33}$$

This time, however, and after some experimentation, we chose the constant

$$\alpha = 0.05\ \text{km} \qquad [k_F \ll m_e]\,, \tag{34}$$

which then fixes

$$\epsilon_0 = 2.488 \times 10^{37}\text{ergs/cm}^3 = 0.01392\ M_\odot c^2/\text{km}^3 \qquad [k_F \ll m_e]\,. \tag{35}$$

Note that this ϵ_0 is much smaller than our choice for the relativistic case. The other constant we need, from Eq. (26), is

$$\beta = 0.005924\ /\text{km}^3 \qquad [k_F \ll m_e]\,, \tag{36}$$

which, unlike the relativistic case, is not larger than α but smaller.

When we first ran our Mathematica code for this case, we (inadvertantly) tried a value of $\bar{p}(0) = 10^{-12}$. This gave us a star with radius R = 5310 km and mass M = 3.131. Oops!, that mass is *bigger* than the largest mass of 1.243 that we found for the relativistic EoS! What did we do wrong?

What happened (and the student can set up her program so this trap can be avoided) is that the choice $\bar{p}(0) = 10^{-12}$ violates the assumption that $k_F \ll m_e$. One really needs values for $\bar{p}(0) < 4 \times 10^{-15}$. This says, in fact, that the relativistic $\bar{p}(0) = 10^{-16}$ case that we plotted in Fig. 3 is not really relativistic.

Results for the non-relativistic case for the last two values of $\bar{p}(0)$ in Table 1 are shown in Table 2. It is quite instructive to compare the differences in the two tables. The masses are, of course, smaller, as expected, and now they vary with $\bar{p}(0)$. Somewhat surprising is that the non-relativistic radius is bigger for $\bar{p}(0) = 10^{-15}$ but smaller for $\bar{p}(0) = 10^{-16}$. Figure 4 shows the pressure distribution for the latter case, to be compared with the corresponding graph in Fig. 3. Note that this pressure curve does not have the peculiar, long flat tail found using the relativistic EoS.

In fact, by this time the student should have realized that neither of these polytropes is very physical, at least not for all cases. The non-relativistic assumption certainly does not work for central pressures $\bar{p}(0) > 10^{-14}$, i.e., for the more massive (and more common) white dwarfs. On the other hand, the relativistic EoS certainly should not work when the pressure becomes small, i.e., in the outer regions of the star (where it

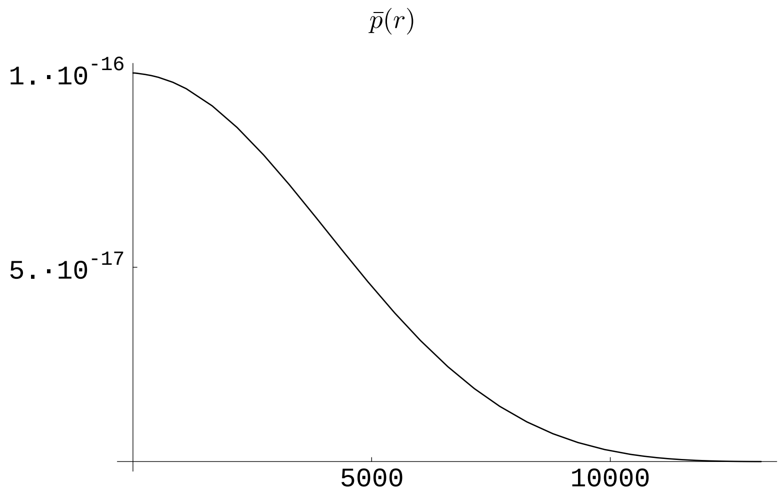

Figure 4: $\bar{p}(r)$ for a white dwarf using the non-relativistic electron Fermi gas model with central pressure $\bar{p}(0) = 10^{-16}$.

eventually goes to zero at the star's radius). So, can one find an EoS to cover the whole range of pressures?

We haven't actually done this for white dwarfs, but the program would be much like that discussed below for the full neutron star. Given the transcendental expressions for energy and pressure that generate the curve shown in Fig. 2, it should be possible to find a fit (using, for example, the built-in fitting function of Mathematica) like

$$\bar{\epsilon}(\bar{p}) = A_{\rm NR}\bar{p}^{3/5} + A_{\rm R}\bar{p}^{3/4}\,. \tag{37}$$

The second term dominates at high pressures (the relativistic case), but the first term takes over for low pressures when the $k_F \gg m_e$ assumption does not hold. (Setting the two terms equal and solving for $\bar{p}$, as Chandrasekhar and Fowler did, gives the value of $\bar{p}$ when special relativity starts to be important.) This expression for $\bar{\epsilon}(\bar{p})$ could then be used in place of the $\bar{p}^{1/\gamma}$ factors on the right hand sides of the structure equations. Proceed to solve numerically as before. We leave this as an exercise for the interested student.

2.4 Pure Neutron Star, Fermi Gas EoS

Having by now become warm, the student can now tackle neutron stars. Here one must include the general relativistic (GR) contributions represented by the three dimensionless factors in the TOV equation, Eq. (5). One of the first things that comes to mind is how one deals numerically with the (apparent) singularities in these factors at $r = 0$ [18].

Also, as in the case of the white dwarfs, there is a question of what to use for the EoS. In this section we show what can be done for *pure* neutron stars, once again using a Fermi gas model for, now, a neutron gas instead of an electron gas. Such a model, however, is unrealistic for two reasons. First, a real neutron star consists not just of neutrons but contains a small fraction of protons and electrons (to inhibit the neutrons from decaying into protons and electrons by their weak interactions). Second,

Table 3: Radius R (in km) and mass M (in $M_\odot$) for pure neutron stars with a non-relativistic Fermi gas EoS.

$\bar{p}(0)$	R (Newton)	M (Newton)	R (GR)	M (GR)
10^{-4}	16.5	0.7747	15.25	0.6026
10^{-5}	20.8	0.3881	20.00	0.3495
10^{-6}	26.3	0.1944	25.75	0.1864

the Fermi gas model ignores the strong nucleon-nucleon interactions, which give important contributions to the energy density. Each of these points will be dealt with in sections below.

2.4.1 The Non-Relativistic Case, $k_F \ll m_n$

For a pure neutron star Fermi gas EoS one can proceed much as in the white dwarf case, substituting the neutron mass m_n for the electron mass m_e in the equations found in Sec. 3. When $k_F \ll m_n$ one finds, again, a polytrope with $\gamma = 5/3$. (More exercises for the student.) The K corresponding to that in Eq. (17) is

$$K_{\rm non-rel} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2 Z}{A m_n c^2}\right)^{5/3} = 6.483 \times 10^{-26}\ \frac{\rm cm^2}{\rm ergs^{2/3}}\,. \tag{38}$$

This time, choosing $\alpha = 1$ km, one finds the scaling factor from Eq. (24) to be

$$\epsilon_0 = 1.603 \times 10^{38}\ {\rm ergs/cm^3} = 0.08969\ {\rm M_\odot} c^2/{\rm km}^3\,. \tag{39}$$

Further, from Eqs. (20) and (26),

$$\bar{K} = 1.914 \quad \text{and } \beta = 0.7636\ /{\rm km}^3\,. \tag{40}$$

Note that, in this case, the constants α and β are of similar size.

Making an estimate of the average energy density of a typical neutron star (mass = $M_\odot$, R = 10 km), one expects that a good starting value for the central pressure $\bar{p}(0)$ to be of order 10^{-4} or less. Our program for this situation is essentially the same as the one for non-relativistic white dwarfs but with appropriate changes of the distance scale. It gives the results shown in Table 3. Note that the GR effects are small, but not negligible, for this non-relativistic EoS. As in the white dwarf case, these are the smaller mass stars. One sees that as the mass gets smaller, the gravitational attraction is less and thus the star extends out to larger radii.

2.4.2 The Relativistic Case, $k_F \gg m_n$

Here there is again a polytropic EoS, but with $\gamma = 1$. In fact, $p = \epsilon/3$, a well-known result for a relativistic gas. The conversion to dimensionless quantities becomes very simple in this case with relationships like $K = \bar{K} = 1/3$. It is still useful to factor out an ϵ_0, which in our program we took to have a value 1.6×10^{38} erg/cm^3, as suggested by the value in the previous sub-section. Then, if we choose this time

$$\alpha = 3R_0 = 4.428\ {\rm km} \tag{41}$$

we find

$$\beta = 3.374\ /{\rm km}^3\,. \tag{42}$$

We expect central pressures $\bar{p}(0)$ in this case to be greater than 10^{-4}. Other than these changes, we wrote a similar program to the one above, taking care to avoid exponents like $1/(\gamma - 1)$.

Running that code gives, at first glance, *enormous* radii, values of R greater than 50 km! We can imagine the student looking frantically for a program bug that isn't there. In fact, what really happens is that, for this EoS, the loop on $\bar{r}$ runs through its whole range, since the pressure $\bar{p}(r)$ never passes through zero. (A plot of $\bar{p}(r)$ looks quite similar, but for distance scale, to that shown in Fig. 3, where $\gamma = 4/5$.) It only falls monotonically toward zero, getting ever smaller. By the time the student recognizes this, she will probably also have realized that the relativistic gas EoS is inappropriate for such small pressures. Something better should be done (as in the next sub-section).

It turns out that the structure equations for $\gamma = 1$ are sufficiently simple that an *analytic* solution for $p(r)$ can be found, which corroborates the above remarks about not having a zero at a finite R. A suggestion for the student is to try a power-law Ansatz.

2.4.3 The Fermi Gas EoS for Arbitrary Relativity

In order to avoid the trap of the relativistic gas, one should find an EoS for the non-interacting neutron Fermi gas which works for all values of the relativity parameter $x = k_F/m_n c$. Taking a hint from the two polytropes, one can try to fit the energy density as a function of pressure, each given as a transcendental function of k_F, with the form

$$\bar{\epsilon}(p) = A_{\rm NR}\bar{p}^{3/5} + A_{\rm R}\bar{p}\,. \tag{43}$$

For low pressures the non-relativistic first term dominates over the second. (The power in the relativistic term is changed from that in Eq. (37).) It is again useful to factor out an ϵ_0 from both ϵ and p. In this case, it is more natural to define it as

$$\epsilon_0 = \frac{m_n^4 c^5}{(3\pi^2\hbar)^3} = 5.346 \times 10^{36}\ \frac{\rm ergs}{\rm cm^3} = 0.003006\ \frac{{\rm M}_\odot c^2}{\rm km^3}\,. \tag{44}$$

Mathematica can easily create a table of exact $\bar{\epsilon}$ and $\bar{p}$ values as a function of k_F. The dimensionless A-values can then be fit using its built-in fitting function. From our efforts we found, to an accuracy of better than 1% over most of the range of k_F [19],

$$A_{\rm NR} = 2.4216\ , \quad A_{\rm R} = 2.8663\,. \tag{45}$$

We used the fitted functional form for $\bar{\epsilon}$ of Eq. (43) in a Mathematica program similar to that for the neutron star based on the non-relativistic EoS. With the ϵ_0 of Eq. (44) and choosing $\alpha = R_0 = 1.476$ km, we obtain $\beta = 0.03778$. Our results for a starting value of $\bar{p}(0) = 0.01$, clearly in the relativistic regime, are

$$R = 15.0\ , \quad M = 1.037\,, \quad \text{Newtonian equations} \tag{46}$$

$$R = 13.4\ , \quad M = 0.717\,, \quad \text{full TOV equation}\,. \tag{47}$$

For this more massive star, the GR effects are significant (as should be expected from the size of GM/c^2R, about 10% in this case). Figure 5 displays the differences.

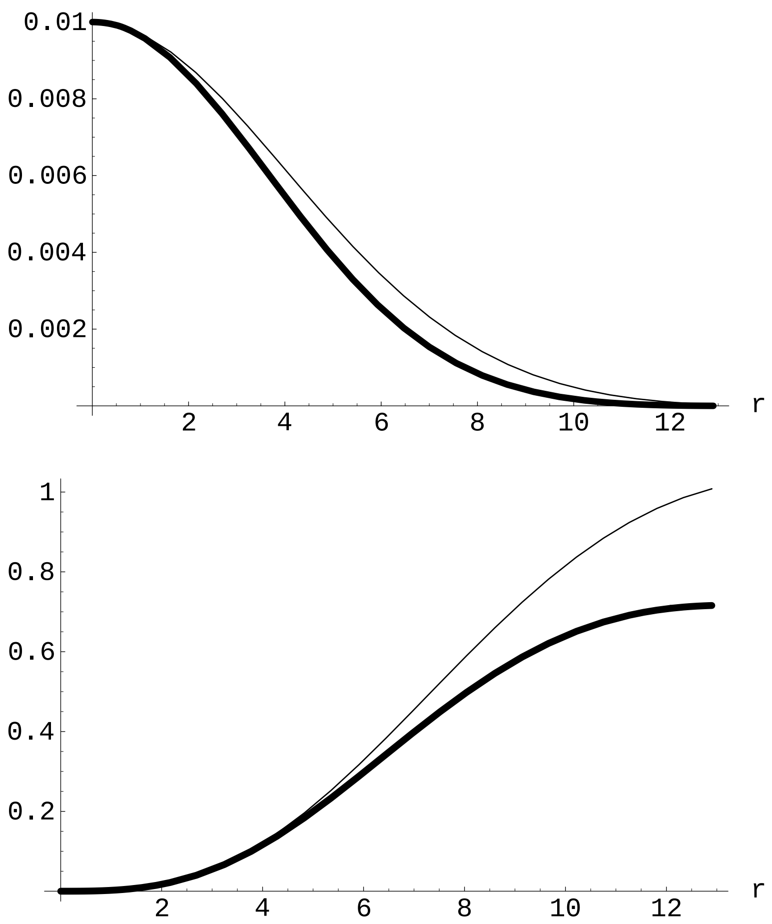

Figure 5: $\bar{p}(r)$ and $\bar{\mathcal{M}}(r)$ (r in km) for a pure neutron star with central pressure $\bar{p}(0) = 0.01$, using a Fermi gas EoS fit valid for all values of k_F. The thin curves are results from the classical Newtonian structure equations, while the thick ones include general relativistic corrections.

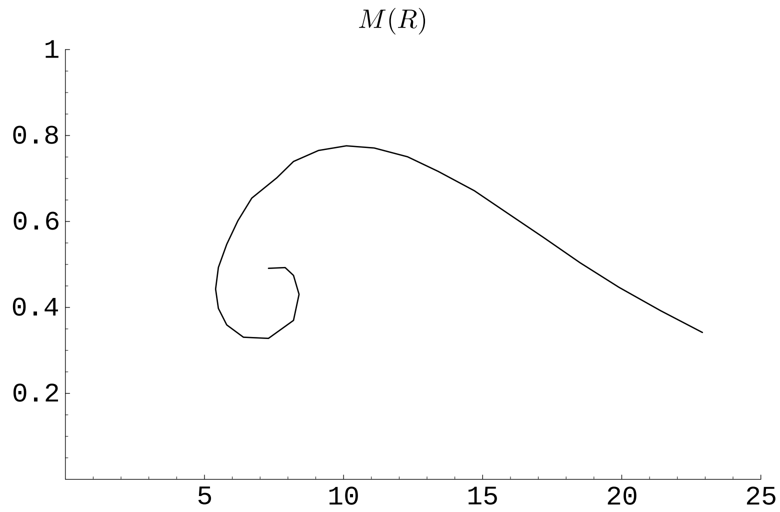

Figure 6: The mass M (in $M_\odot$) and radius R (in km) for pure neutron stars, using a Fermi gas EoS. The stars of low mass and large radius are solutions of the TOV equations for small values of central pressure $\bar{p}(0)$. The stars to the right of the maximum at $R = 11$ are stable, while those to the left will suffer gravitational collapse.

It is now instructive to make a long run of calculations for a range of $\bar{p}(0)$ values. We display in Fig. 6 a (parametric) plot of M and R as they depend on the central pressure. The low-mass/large-radius stars are to the right in the graph and correspond to small starting values of $\bar{p}(0)$. As the central pressure increases, the total mass that the star can support increases. And, the bigger the star mass, the bigger the gravitational attraction which draws in the periphery of the star, making stars with smaller radii. That is, increasing $\bar{p}(0)$ corresponds to "climbing the hill," moving upward and to the left in the diagram.

At about $\bar{p}(0) = 0.03$ one gets to the top of the hill, achieving a maximum mass of about 0.8 $M_\odot$ at a radius $R \approx 11$ km. That maximum M and its R agree with Oppenheimer and Volkov's seminal 1939 result for a Fermi gas EoS.

What about the solutions in Fig. 6 that are "over the hill," i.e., to the left of the maximum? It turns out that these stars are unstable against gravitational collapse into a black hole. The question of stability, however, is a complicated issue [20], perhaps too difficult for a student at this level. The fact that things collapse to the left of the maximum, however, means that one probably shouldn't worry too much about the peculiar curly-cue tail to the M-R curve in the figure. It appears to be an artifact for *very* large values of $\bar{p}(0)$, also seen in other calculations, even though it is especially prominent for this Fermi gas EoS.

2.4.4 Why Is There a Maximum Mass?

On general grounds one can argue that cold compact objects such as white dwarfs and neutron stars must possess a limiting mass beyond which stable hydrostatic configurations are not possible. This limiting mass is often called the maximum mass of the object and was briefly mentioned in the discussion at the end of Sec. 2.2 and that relating to Fig. 6. In what follows, we outline the general argument.

The thermal component of the pressure in cold stars is by definition negligible. Thus, variations in both the energy density and pressure are only caused by changes in the density. Given this simple observation, let us examine why we expect a maximum mass in the Newtonian case.

Here, an increase in the density results in a proportional increase in the energy density. This results in a corresponding increase in the gravitational attraction. To balance this, we require that the increment in pressure is large enough. However, the rate of change of pressure with respect to energy density is related to the speed of sound (see Sec. 6.3). In a purely Newtonian world, this is in principle unbounded. However, the speed of all propagating signals cannot exceed the speed of light. This then puts a bound on the pressure increment associated with changes in density.

Once we accept this bound, we can safely conclude that all cold compact objects will eventually run into the situation in which any increase in density will result in an additional gravitational attraction that cannot be compensated for by the corresponding increment in pressure. This leads naturally to the existence of a limiting mass for the star.

When we include general relativistic corrections, as discussed in Sec. 2.2 earlier, they act to "amplify" gravity. Thus we can expect the maximum mass to occur at a somewhat lower mass than in the Newtonian case.

2.5 Neutron Stars with Protons and Electrons, Fermi Gas EoS

As mentioned at the beginning of the last section, neutron stars are not made only of neutrons. There must also be a small fraction of protons and electrons present. The reason for this is that a free neutron will undergo a weak decay,

$$n \to p + e^- + \bar{\nu}_e \;, \tag{48}$$

with a lifetime of about 15 minutes. So, there must be something that prevents this decay in the case of the star, and that is the presence of the protons and electrons.

The decay products have low energies ($m_n - m_p - m_e = 0.778$ MeV), with most of that energy being carried away by the light electron and (nearly massless) neutrino [21]. If all the available low-energy levels for the decay proton are already filled by the protons already present, then the Pauli exclusion principle takes over and prevents the decay from taking place.

The same might be said about the presence of the electrons, but in any case the electrons must be present within the star to cancel the positive charge of the protons. A neutron star is electrically neutral. We saw earlier that the number density of a particle species is fixed in terms of that particle's Fermi momentum [see Eq. (7)]. Thus equal numbers of electrons and protons implies that

$$k_{F,p} = k_{F,e}\,. \tag{49}$$

In addition to charge neutrality, we also require weak interaction equilibrium, i.e., as many neutron decays [Eq. (48)] taking place as electron capture reactions, $p + e^- \to n + \nu_e$. This equilibrium can be expressed in terms of the chemical potentials for the three particle species,

$$\mu_n = \mu_p + \mu_e \,. \tag{50}$$

We already defined the chemical equilibrium for a particle in Sec. 3.2 after Eq. (13),

$$\mu_i(k_{F,i}) = \frac{d\epsilon}{dn} = (k_{F,i}^2 + m_i^2)^{1/2} \,, \quad i = n, p, e \,. \tag{51}$$

where, for the time being, we have set $c = 1$ to simplify the equations somewhat. (The student is urged to prove the right-hand equality.) From Eqs. (49), (50), and (51) we can find a constraint determining $k_{F,p}$ for a given $k_{F,n}$,

$$(k_{F,n}^2 + m_n^2)^{1/2} - (k_{F,p}^2 + m_p^2)^{1/2} - (k_{F,p}^2 + m_e^2)^{1/2} = 0 \,. \tag{52}$$

While an ambitious algebraist can probably solve this equation for $k_{F,p}$ as a function of $k_{F,n}$, we were somewhat lazy and let Mathematica do it, finding

$$k_{F,p}(k_{F,n}) = \frac{[(k_{F,n}^2 + m_n^2 - m_e^2)^2 - 2m_p^2(k_{F,n}^2 + m_n^2 + m_e^2) + m_p^4]^{1/2}}{2(k_{F,n}^2 + m_n^2)^{1/2}} \tag{53}$$

$$\approx \frac{k_{F,n}^2 + m_n^2 - m_p^2}{2(k_{F,n}^2 + m_n^2)^{1/2}} \quad \text{for } \frac{m_e}{k_{F,n}} \to 0 \,. \tag{54}$$

The total energy density is the sum of the individual energy densities,

$$\epsilon_{\text{tot}} = \sum_{i=n,p,e} \epsilon_i \,, \tag{55}$$

where

$$\epsilon_i(k_{F,i}) = \int_0^{k_{F,i}} (k^2 + m_i)^{1/2} k^2 dk = \epsilon_0 \, \bar{\epsilon}_i(x_i, y_i) \,, \tag{56}$$

and, as before [22],

$$\epsilon_0 = m_n^4 / 3\pi^2 \hbar^3 \,, \tag{57}$$

$$\bar{\epsilon}_i(x_i, y_i) = \int_0^{x_i} (u^2 + y_i^2)^{1/2} u^2 du \,, \tag{58}$$

$$x_i = k_{F,i}/m_i \,, \; y_i = m_i/m_n \,. \tag{59}$$

The corresponding total pressure is

$$p_{\text{tot}} = \sum_{i=n,p,e} p_i \,, \tag{60}$$

$$p_i(k_{F,i}) = \int_0^{k_{F,i}} (k^2 + m_i)^{-1/2} k^4 dk = \epsilon_0 \, \bar{p}_i(x_i, y_i) \,, \tag{61}$$

$$\bar{p}_i(x_i, y_i) = \int_0^{x_i} (u^2 + y_i^2)^{-1/2} u^4 du \,. \tag{62}$$

Using Mathematica the (dimensionless) integrals can be expressed in terms of log and $\sinh^{-1}$ functions of x_i and y_i. One can then generate a table of $\bar{\epsilon}_{\text{tot}}$ versus $\bar{p}_{\text{tot}}$ values for an appropriate range of $k_{F,n}$'s. This, in turn, can be fitted to the same form of two terms as before in Eq. (43). We found, this time, the coefficients to be

$$A_{\text{NR}} = 2.572 \,, \quad A_{\text{R}} = 2.891 \,. \tag{63}$$

These coefficients are not much changed from those in Eq. (45) for the pure neutron star. Therefore, we expect that the M versus R diagram for this more realistic Fermi gas model would not be much different from that in Fig. 6.

2.6 Introducing Nuclear Interactions

Nucleon-nucleon interactions can be included in the EoS (they are important) by constructing a simple model for the nuclear potential that reproduces the general features of (normal) nuclear matter. In doing so we were much guided by the lectures of Prakash [6].

We will use MeV and fm (10^{-13} cm) as our energy and distance units for much of this section, converting back to $M_\odot$ and km later. We will also continue setting $c = 1$ for now. In this regard, the important number to remember for making conversions is $\hbar c = 197.3$ MeV-fm. We will also neglect the mass difference between protons and neutrons, labeling their masses as m_N.

The Bethe-Weizäcker mass formula [23] for the binding energy of nuclides with Z protons and N neutrons (mass number $A = N + Z$) reads

$$BE = E_{\rm Vol}A - E_{\rm Surf}A^{2/3} - E_{\rm Sym}\frac{(N-Z)^2}{A} - \frac{3}{5}\frac{Z(Z-1)e^2}{4\pi\varepsilon_0 R_A} + E_{\rm Pair}, \quad (64)$$

where the volume contribution to the binding energy pro nucleon is the dominant one in the limit of infinite nuclear matter $\lim_{A\to\infty} E_{\rm Vol} = (E/A - m_N)|_{n_0} = 16$ MeV. The suface energy is $E_{\rm Surf} = 17$ MeV, the symmetry energy is $E_{\rm Sym} = 30$ MeV and for the pairing energy there are three possibilities $E_{\rm Pair} = \{\Delta, 0, -\Delta\}$ for {even-even, even-odd, odd-odd} nuclei, respectively. The pairing energy gap is $\Delta = 25\ A^{-1}$ MeV. The remaining term in (64) is the Coulomb energy, which depends on the nuclear charge number Z and the nuclear radius $R_A = 1.24\ A^{1/3}$ fm. For normal symmetric nuclear matter ($N = Z$), an equilibrium number density n_0 of 0.16 nucleons/fm^3 is obtained. For this value of n_0 the Fermi momentum is $k_F^0 = 263$ MeV/c [see Eq. (7)]. This momentum is small enough compared with $m_N = 939$ MeV/c^2 to allow a non-relativistic treatment of normal nuclear matter. At this density, the average binding energy per nucleon, $BE = E/A - m_N$, is -16 MeV. These are two physical quantities we definitely want our nuclear potential to respect, but there are two more that we'll need to fix the parameters of the model.

We chose one of these as the *nuclear compressibility*, K_0, to be defined below. This is a quantity which is not all that well established but is in the range of 200 to 400 MeV. The other is the so-called *symmetry energy* term, which, when $Z = 0$, contributes about 30 MeV of energy above the symmetric matter minimum at n_0. (This quantity might really be better described as an asymmetry parameter, since it accounts for the energy that comes in when $N \neq Z$.)

2.6.1 Symmetric Nuclear Matter

We defer the case when $N \neq Z$, which is our main interest in this paper, to the next sub-section. Here we concentrate on getting a good (enough) EoS for nuclear matter when $N = Z$, or, equivalently, when the proton and neutron number densities are equal, $n_n = n_p$. The total nucleon density $n = n_n + n_p$.

We need to relate the first three nuclear quantities, n_0, BE, and K_0 to the energy density for symmetric nuclear matter, $\epsilon(n)$. Here $n = n(k_F)$ is the nuclear density (at and away from n_0). We will not worry in this section about the electrons that are present, since, as was seen in the last section, its contribution is small. The energy density now will include the nuclear potential, $V(n)$, which we will model below in terms of two simple functions with three parameters that are fitted to reproduce the above three nuclear quantities. [The fourth quantity, the symmetry energy, will be used in the next sub-section to fix a term in the potential which is proportional to $(N - Z)/A$.]

First, the average energy per nucleon, E/A, for symmetric nuclear matter is related to ϵ by

$$E(n)/A = \epsilon(n)/n \,, \tag{65}$$

which includes the rest mass energy, m_N and has dimensions of MeV. As a function of n, $E(n)/A - m_N$ has a minimum at $n = n_0$ with a depth $BE = -16$ MeV. This minimum occurs when

$$\frac{d}{dn}\left(\frac{E(n)}{A}\right) = \frac{d}{dn}\left(\frac{\epsilon(n)}{n}\right) = 0 \ \text{ at } n = n_0 \,. \tag{66}$$

This is one constraint of the parameters of $V(n)$. Another, of course, is the binding energy,

$$\frac{\epsilon(n)}{n} - m_N = BE \ \text{ at } n = n_0 \,. \tag{67}$$

The positive curvature at the bottom of this valley is related to the nuclear (in)compressibility by [24]

$$K(n) = 9\frac{dp(n)}{dn} = 9\left[n^2\frac{d^2}{dn^2}\left(\frac{\epsilon}{n}\right) + 2n\frac{d}{dn}\left(\frac{\epsilon}{n}\right)\right] \,, \tag{68}$$

using Eq. (13), which defines the pressure in terms of the energy density. At $n = n_0$ this quantity equals K_0. (The factor of 9 is a historical artifact from the convention originally defining K_0.)

(Question: why does one *not* have to calculate the pressure at $n = n_0$?)

The $N = Z$ potential in $\epsilon(n)$ we will model as [6]

$$\frac{\epsilon(n)}{n} = m_N + \frac{3}{5}\frac{\hbar^2 k_F^2}{2m_N} + \frac{A}{2}u + \frac{B}{\sigma+1}u^\sigma \,, \tag{69}$$

where $u = n/n_0$ and σ are dimensionless and A and B have dimensions of MeV. The first term represents the rest mass energy and the second the average kinetic energy per nucleon. [These two terms are leading in the non-relativistic limit of the nucleonic version of Eq. (10).] For $k_F(n_0) = k_F^0$ we will abbreviate the kinetic energy term as $\langle E_F^0 \rangle$, which evaluates to 22.1 MeV. The kinetic energy term in Eq. (69) can be better written as $\langle E_F^0 \rangle u^{2/3}$.

From the above three constraints, Eqs. (66)-(68), and noting that $u = 1$ at $n = n_0$, we get three equations for the parameters A, B, and σ:

$$\langle E_F^0 \rangle + \frac{A}{2} + \frac{B}{\sigma+1} = BE \,, \tag{70}$$

$$\frac{2}{3}\langle E_F^0 \rangle + \frac{A}{2} + \frac{B\sigma}{\sigma+1} = 0 \,, \tag{71}$$

$$\frac{10}{9}\langle E_F^0 \rangle + A + B\sigma = \frac{K_0}{9} \,. \tag{72}$$

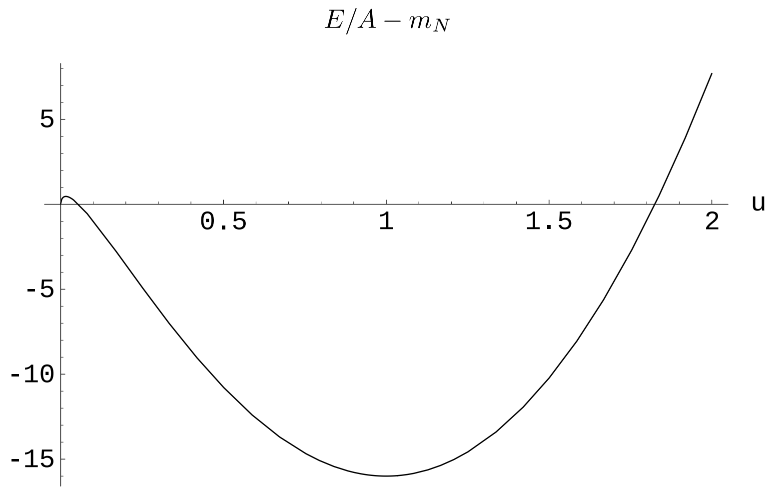

Figure 7: The average energy per nucleon, less its rest mass, as a function of $u = n/n_0$ (in MeV). The position of the minimum is at $n = n_0 = 0.16\ \mathrm{fm}^{-3}$, its depth there is $BE = -16$ MeV, and its curvature (second derivative) there corresponds to nuclear compressibility $K_0 = 400$ MeV.

Solving these equations (which we found easier to do by hand than with Mathematica), one finds

$$\sigma = \frac{K_0 + 2\langle E_F^0\rangle}{3\langle E_F^0\rangle - 9BE}, \tag{73}$$

$$B = \frac{\sigma+1}{\sigma-1}\left[\frac{1}{3}\langle E_F^0\rangle - BE\right], \tag{74}$$

$$A = BE - \frac{5}{3}\langle E_F^0\rangle - B\,. \tag{75}$$

Numerically, for $K_0 = 400$ MeV (which is perhaps a high value),

$$A = -122.2 \text{ MeV}, \quad B = 65.39 \text{ MeV}, \quad \sigma = 2.112\,. \tag{76}$$

Note that $\sigma > 1$, a point we will come back to below, since it violates a basic principle of physics called "causality."

The student can try other values of K_0 to see how the parameters A, B, and σ change. More interesting is to see how the interplay between the A- and B-terms gives the valley at $n = n_0$. Figure 7 shows $E/A - m_N$ as a function of n using the parameters of Eq. (76). We would hope the student notices the funny little positive bump in this plot near $n = 0$ and sorts out the reason for its occurrence.

Given $\epsilon(n)$ from Eq. (69) one can find the pressure using Eq. (13),

$$p(n) = n^2 \frac{d}{dn}\left(\frac{\epsilon}{n}\right) = n_0\left[\frac{2}{3}\langle E_F^0\rangle u^{5/3} + \frac{A}{2}u^2 + \frac{B\sigma}{\sigma+1}u^{\sigma+1}\right]. \tag{77}$$

For the parameters of Eq. (76) its dependence on n is shown in Fig. 8. On first seeing this graph, the student should wonder why $p(u=1) = p(n_0) = 0$. And, what is the meaning of the negative values for pressure below $u = 1$? (Hint: what is "cavitation"?)

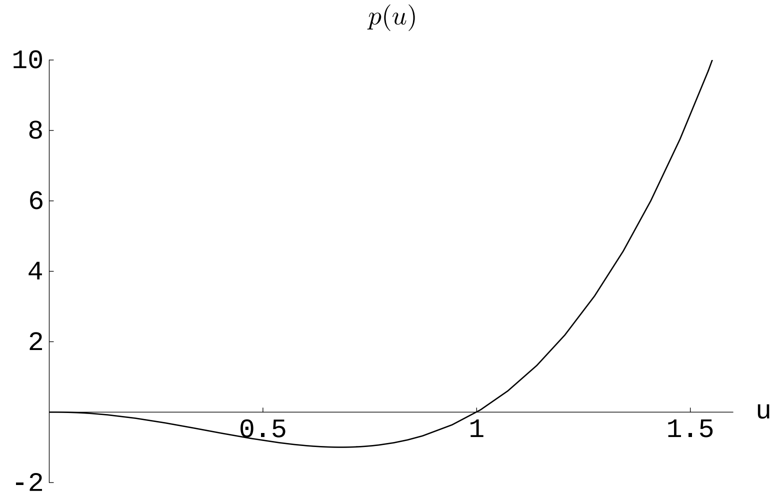

Figure 8: The pressure for symmetric nuclear matter as a function of $u = n/n_0$. The student should ask what it means when the pressure is negative and why it is 0 at $u = 1$.

So, if this $N = Z$ case were all we had for the nuclear EoS, a plot of $\epsilon(n)$ versus $p(n)$ would only make sense for $n \geq n_0$. Such a plot looks much like a parabola opening to the right for the range $0 < u < 3$. At very large values of u, however, $\epsilon \approx p/3$, as it should for a relativistic nucleon gas (see Section 4.2). We don't pursue this symmetric nuclear matter EoS further since our interest is in the case when $N \gg Z$ [25].

2.6.2 Non-Symmetric Nuclear Matter

We continue following Prakash's notes [6] closely. Let us represent the neutron and proton densities in terms of a parameter α as

$$n_n = \frac{1+\alpha}{2}\, n\,, \quad n_p = \frac{1-\alpha}{2}\, n\,. \tag{78}$$

This α is not to be confused with the constant defined in Eq. (23). For pure neutron matter $\alpha = 1$. Note that

$$\alpha = \frac{n_n - n_p}{n} = \frac{N-Z}{A}\,, \tag{79}$$

so we can expect that the isospin-symmetry-breaking interaction is proportional to α (or some power of it). An alternative notation is in terms of the fraction of protons in the star,

$$x = \frac{n_p}{n} = \frac{1-\alpha}{2}\,. \tag{80}$$

We now consider how the energy density changes from the symmetric case discussed above, where $\alpha = 0$ (or $x = 1/2$).

First, there are contributions to the kinetic energy part of ϵ from both neutrons and protons,

$$\epsilon_{KE}(n,\alpha) \quad = \quad \frac{3}{5}\frac{k_{F,n}^2}{2m_N}\, n_n + \frac{3}{5}\frac{k_{F,p}^2}{2m_N}\, n_p$$

$$= \quad n \langle E_F \rangle \frac{1}{2} \left[(1+\alpha)^{5/3} + (1-\alpha)^{5/3} \right] , \tag{81}$$

where

$$\langle E_F \rangle = \frac{3}{5} \frac{\hbar^2}{2m_N} \left(\frac{3\pi^2 n}{2} \right)^{2/3} \tag{82}$$

is the mean kinetic energy of symmetric nuclear matter at density n. For $n = n_0$ we note that $\langle E_F \rangle = 3 \langle E_F^0 \rangle /5$ [see Eq. (69)]. For non-symmetric matter, $\alpha \neq 0$, the excess kinetic energy is

$$\begin{aligned} \Delta\epsilon_{KE}(n,\alpha) &= \epsilon_{KE}(n,\alpha) - \epsilon_{KE}(n,0) \\ &= n \langle E_F \rangle \left\{ \frac{1}{2} \left[(1+\alpha)^{5/3} + (1-\alpha)^{5/3} \right] - 1 \right\} \\ &= n \langle E_F \rangle \left\{ 2^{2/3} \left[(1-x)^{5/3} + x^{5/3} \right] - 1 \right\} . \end{aligned} \tag{83}$$

For pure neutron matter, $\alpha = 1$,

$$\Delta\epsilon_{KE}(n,\alpha) = n \langle E_F \rangle \left(2^{2/3} - 1 \right) . \tag{84}$$

It is also useful to expand to leading order in α,

$$\Delta\epsilon_{KE}(n,\alpha) = n \langle E_F \rangle \frac{5}{9} \alpha^2 \left(1 + \frac{\alpha^2}{27} + \cdots \right) \tag{85}$$

$$= n \, E_F \, \frac{\alpha^2}{3} \left(1 + \frac{\alpha^2}{27} + \cdots \right) . \tag{86}$$

Keeping terms to order α^2 is evidently good enough for most purposes. For pure neutron matter, the energy per particle (which, recall, is ϵ/n) at normal density is $\Delta\epsilon_{KE}(n_0, 1)/n_0 \approx 13$ MeV, more than a third of the total bulk symmetry energy of 30 MeV, our fourth nuclear parameter.

Thus the potential energy contribution to the bulk symmetry energy must be 20 MeV or so. Let us assume the quadratic approximation in α also works well enough for this potential contribution and write the total energy per particle as

$$E(n,\alpha) = E(n,0) + \alpha^2 S(n) , \tag{87}$$

The isospin-symmetry breaking is proportional to α^2, which reflects (roughly) the pair-wise nature of the nuclear interactions.

We will assume $S(u)$, $u = n/n_0$, has the form

$$S(u) = (2^{2/3} - 1) \frac{3}{5} \langle E_F^0 \rangle \left(u^{2/3} - F(u) \right) + S_0 F(u) . \tag{88}$$

Here $S_0 = 30$ MeV is the bulk symmetry energy parameter. The function $F(u)$ must satisfy $F(1) = 1$ [so that $S(u=1) = S_0$] and $F(0) = 0$ [so that $S(u=0) = 0$; no matter means no energy]. Besides these two constraints there is, from what we presently know, a lot of freedom in what one chooses for $F(u)$. We will make the simplest possible choice here, namely,

$$F(u) = u , \tag{89}$$

but we encourage the student to try other forms satisfying the conditions on $F(u)$, such as $\sqrt{u}$, to see what difference it makes.

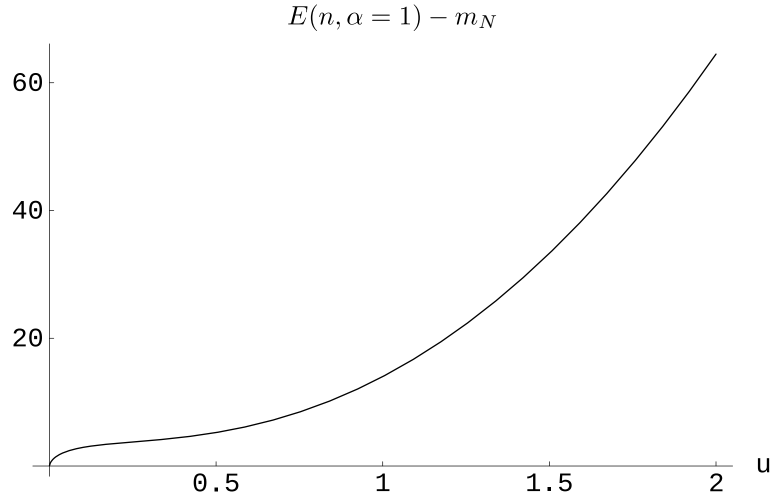

Figure 9: The average energy per neutron (less its rest mass), in MeV, for pure neutron matter, as a function of $u = n/n_0$. The parameters for this curve are for a nuclear compressibility K_0 of 400 MeV.

Figure 9 shows the energy per particle for pure neutron matter, $E(n, 1) - m_N$, as a function of u for the parameters of Eq. (76) and $S_0 = 30$ MeV. In contrast with the $\alpha = 0$ plot in Fig. 7, $E(n, 1) \geq 0$ and is monotonically increasing. The plot looks almost quadratic as a function of u. The dominant term at large u goes like u^σ, and $\sigma = 2.112$ (for this case). However, one might have expected a *linear* increase instead. We will return to this point in Sec. 6.3.

Given the energy density, $\epsilon(n, \alpha) = n_0 u E(n, \alpha)$, the corresponding pressure is, from Eq. (13),

$$\begin{aligned} p(n, x) &= u \frac{d}{du} \epsilon(n, \alpha) - \epsilon(n, \alpha) \\ &= p(n, 0) + n_0 \alpha^2 \left[\frac{2^{2/3} - 1}{5} \langle E_F^0 \rangle \left(2u^{5/3} - 3u^2 \right) + S_0 u^2 \right] \end{aligned} \tag{90}$$

where $p(n, 0)$ is defined by Eq. (77). Figure 10 shows the dependence of the pure neutron $p(n, 1)$ and $\epsilon(n, 1)$ on $u = n/n_0$, ranging from 0 to 10 times normal nuclear density. Both functions increase smoothly and monotonically from $u = 0$. We hope the student would wonder why the pressure becomes greater than the energy density around $u = 6$. Why doesn't it go like a relativistic nucleon gas, $p = \epsilon/3$? (Hint: check the assumptions.)

One can now look at the EoS, i.e., the dependance of p on ϵ (the points in Fig. 11). The pressure is smooth, non-negative, and monotonically increasing as a function of ϵ. In fact it looks almost quadratic over this energy range ($0 \leq u \leq 5$). This suggests that it might be reasonable to see if one can make a simple, polytropic fit. If we try that using a form

$$p(\epsilon) = \kappa_0 \, \epsilon^\gamma \, , \tag{91}$$

we find the fit shown in Fig. 11 as the solid curve with

$$\kappa_0 = 3.548 \times 10^{-4} \, , \quad \gamma = 2.1 \, , \tag{92}$$

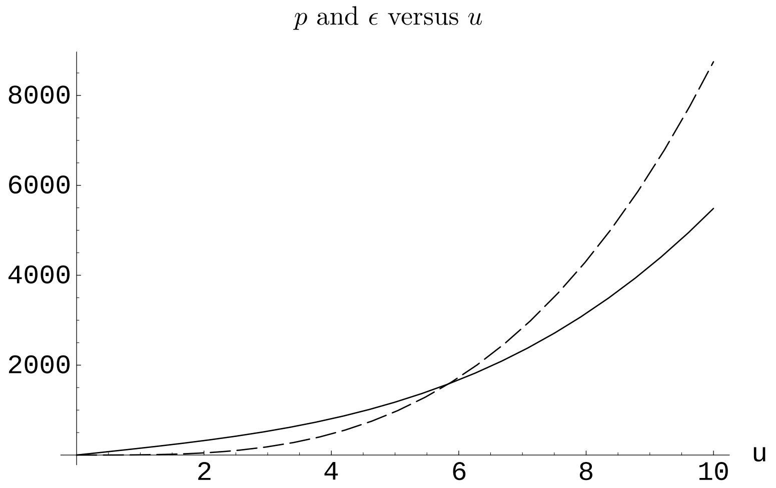

Figure 10: The pressure (dashed curve) and energy density (solid) for pure neutron matter, as a function of $u = n/n_0$. Units for the y-axis are MeV/fm^3. This curve uses parameters based on a nuclear compressibility $K_0 = 400$ MeV.

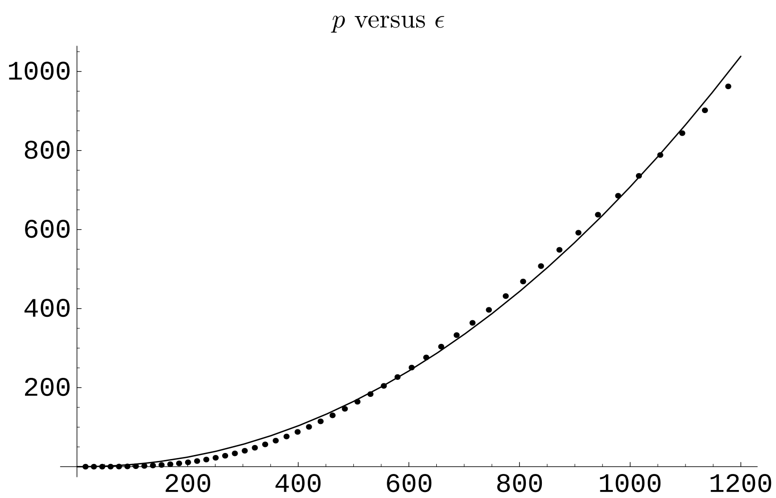

Figure 11: The equation of state for pure neutron matter ($\alpha = 1$), i.e., the dependence of pressure (y-axis) versus energy density (x-axis). Units for both axes are MeV/fm^3, and the nuclear compressibility in this case is $K_0 = 400$ MeV. The points are values calculated directly from Eq. (87), multiplied by n, and Eq. (90), while the solid curve is a fit to these points given in Eqs. (91) and (92).

where κ_0 has appropriate units so that p and ϵ are in MeV/fm^3. (We simply guessed and set γ to that value.)

This polytrope can now be used in solving the TOV equation for a pure neutron star with nuclear interactions. Alternatively, one might solve for the structure by using the functional forms from Eq. (87), multiplied by n, and Eq. (90) directly. We defer that for a bit, since it would be a good idea to first find an EoS which doesn't violate causality, a basic tenet of special relativity.

2.6.3 Does the Speed of Sound Exceed That of Light?

What is the speed of sound in nuclear matter? Starting from the elementary formula for the square of the speed of sound in terms of the bulk modulus [26], one can show that

$$\left(\frac{c_s}{c}\right)^2 = \frac{B}{\rho c^2} = \frac{dp}{d\epsilon} = \frac{dp/dn}{d\epsilon/dn}\,. \tag{93}$$

To satisfy relativistic causality we must require that the sound speed does not exceed that of light. This can happen when the density becomes very large, i.e., when $u \to \infty$. For the simple model of nuclear interactions presented in the last section, the dominant terms at large u in p and ϵ are those going like $u^{\sigma+1}$. Thus, from Eq. (87), multiplied by n, and Eq. (90), we see that

$$\left(\frac{c_s}{c}\right)^2 = \frac{dp/dn}{d\epsilon/dn} \to \sigma = 2.11 \tag{94}$$

for the parameters of Eq. (76), and indeed for any set of parameters with K_0 greater than about 180 MeV.

One can recover causality (i.e., speeds of sound less than light) by assuring that both $\epsilon(u)$ and $p(u)$ grow no faster than u^2. There must still be an interplay between the A- and B-terms in the nuclear potential, but one simple way of doing this is to modify the B-term by introducing a fourth parameter C so that, for symmetric nuclear matter ($\alpha = 0$),

$$V_{\mathrm{Nuc}}(u, 0) = \frac{A}{2}u + \frac{B}{\sigma+1}\;\frac{u^\sigma}{1 + Cu^{\sigma-1}}\,. \tag{95}$$

One can choose C small enough so that the effect of the denominator only becomes appreciable for very large u. The presence of the denominator would modify and complicate the constraint equations for A, B, and σ from those given in Eqs. (70)–(72). However, for small C, which can be chosen as one wishes, the values for the other parameters should not be much changed from those, say, in Eq. (76). Thus, with a little bit of trial and error, one can simply readjust the A, B, and σ values to put the minimum of $E/A - m_N$ at the right position (n_0) and depth (BE), hoping that the resulting value of the (poorly known) compressibility K_0 remains sensible.

In our calculations we chose $C = 0.2$ and started the hand search with the $K_0 = 400$ MeV parameters in Eq. (76). We found that, by fiddling only with B and σ, we could re-fit n_0 and B with only small changes,

$$B = 65.39 \to 83.8 \text{ MeV}\,, \quad \sigma = 2.11 \to 2.37\,, \tag{96}$$

somewhat larger than before. For these new values of B and σ, A changes from -122.2 MeV to -136.7 MeV, and K_0 from 400 to 363.2 MeV. That is, it remains a reasonable nuclear model.

One can now proceed as in the last section to get $\epsilon(n,\alpha)$, $p(n,\alpha)$, and the EoS, $p(\epsilon,\alpha)$. The results are not much different from those shown in the figures of the previous sub-section. This time we decided to live with a quadratic fit for the EoS for pure neutron matter, finding

$$p(\epsilon, 1) = \kappa_0 \epsilon^2 \,, \quad \kappa_0 = 4.012 \times 10^{-4} \,. \tag{97}$$

This is not much different from before, Eq. (92). Somewhat more useful for solving the TOV equation is to express ϵ in terms of p,

$$\epsilon(p) = (p/\kappa_0))^{1/2} \,. \tag{98}$$

2.6.4 Pure Neutron Star with Nuclear Interactions

Having laid all this groundwork, the student can now proceed to solve the TOV equations as before for a pure neutron star, using the fit for $\epsilon(p)$ found in the previous sub-section. It is, once again, useful to convert from the units of MeV/fm^3 to ergs/cm^3 to $M_\odot$/km^3 and dimensionless $\bar{p}$ and $\bar{\epsilon}$. By now the student has undoubtedly grown quite accustomed to that procedure.

$$\bar{\epsilon}(\bar{p}) = (\kappa_0 \epsilon_0)^{-1/2} \bar{p}^{1/2} = A_0 \bar{p}^{1/2} \,, \quad A_0 = 0.8642 \,, \tag{99}$$

where this time we defined

$$\epsilon_0 = \frac{m_n^4 c^5}{3\pi^2 \hbar^3} \,. \tag{100}$$

With this, the constant α that occurs on the right-hand side of the TOV equation, Eq. (22), is $\alpha = A_0 R_0 = 1.276$ km. The constant for the mass equation, Eq. (25), is $\beta = 0.03265$, again in units of 1/km^3.

Now proceeding as before, one can solve the coupled TOV equations for $\bar{p}(r)$ and $\bar{\mathcal{M}}(r)$ for various initial central pressures, $\bar{p}(0)$. We don't exhibit here plots of the solutions, as they look very similar to those for the Fermi gas EoS, Fig. 5.

More interesting is to solve for a range of initial $\bar{p}(0)$'s, generating, as before, a mass M versus radius R plot which now includes nucleon-nucleon interactions (Fig. 12). The effect of the nuclear potential is enormous, on comparing with the Fermi gas model predictions for M vs. R shown in Fig. 6. The maximum star mass this time is about 2.3 $M_\odot$, rather than 0.8 $M_\odot$. The radius for this maximum mass star is about 13.5 km, somewhat larger than the Fermi gas model radius of 11 km. The large value of maximum M is a reflection of the large value of nuclear (in)compressibility $K_0 = 363$ MeV. The more incompressible something is, the more mass it can support. Had we fit to a smaller value of K_0 we would have gotten a smaller maximum mass.

2.6.5 What About a Cosmological Constant?

We do not know (either) if there is one, but there are definite indications that a great part of the make-up of our universe is something called "Dark Energy" [27]. This conclusion comes about because we have recently learned that something, at the present time, is causing the universe to be accelerating, instead of slowing down (as would be expected after the Big Bang).

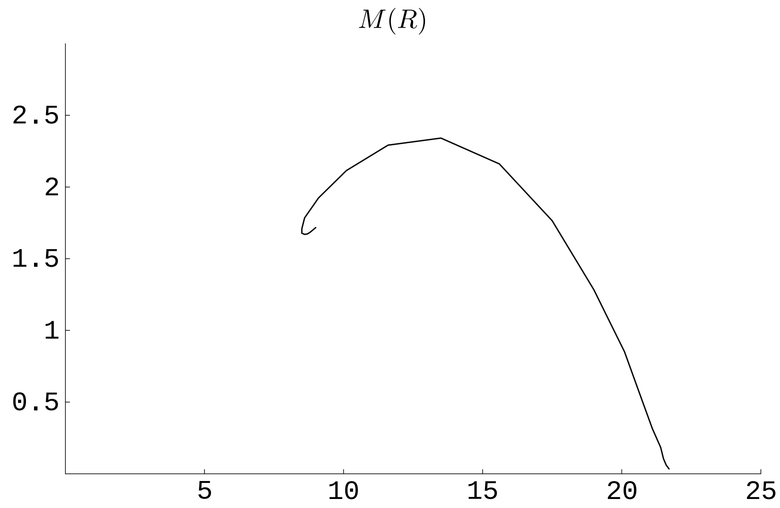

Figure 12: The mass M and radius R for pure neutron stars using an EoS which contains nucleon-nucleon interactions. Only those stars to the right of the maximum are stable against gravitational collapse. Compare this graph with that in Fig. 6 which is based on a non-interacting Fermi gas model for the EoS.

One way (of several) to interpret this dark energy is as Einstein's cosmological constant, which contributes a term $\Lambda g_{\mu\nu}$ to the right-hand side of Einstein's field equation, the basic equation of general relativity. The most natural value for Λ would be zero, but that may not be the way the world is. If Λ *is* non-zero, it is nonetheless surprisingly small.

What would the effect of a non-zero cosmological constant be for the structure of a neutron star? It turns out that the only modification to the TOV equation is in the correction factor

$$\left[1 + \frac{4\pi r^3 p(r)}{\mathcal{M}(r)c^2}\right] \to \left[1 + \frac{4\pi r^3 p(r)}{\mathcal{M}(r)c^2} - \frac{\Lambda r^3}{2G\mathcal{M}(r)}\right] . \tag{101}$$

So, we encourage the student to, first, understand the units of Λ and then to see what values for it might affect the structure of a typical neutron star.

2.7 Conclusions

The materials we have described in this paper would be quite suitable as an undergraduate thesis or special topics course accessible to a junior or senior physics major. It is a topic rich in the subjects the student will have covered in his or her courses, ranging from thermodynamics to quantum statistics to nuclear physics.

The major emphasis in such a project is on constructing a (simple) equation of state. This is needed to be able to solve the non-linear structure equations. Solving those equations numerically, of course, develops the student's computational skills. Along the way, however, he or she will also learn some of the lore regarding degenerate stars, e.g., white dwarfs and neutron stars. And, in the latter case, the student will also come to appreciate the relative importance of special and general relativity.

References

[1] It is widely beleived that neutron stars were proposed by Lev Landau in 1932, very soon after the neutron was discovered (although we are not aware of any documented proof of this). In 1934 Fritz Zwicky and Walter Baade speculated that they might be formed in Type II supernova explosions, which is now generally accepted as true.

[2] An on-line catalog of pulsars can be found at http:/pulsar.princeton.edu.

[3] An intermediate-level on-line tutorial on the physics of pulsars can be found at http://www.jb.man.ac.uk/research/pulsar. This tutorial follows the book by Andrew G. Lyne and Francis Graham-Smith, *Pulsar Astronomy*, 2nd. ed., Cambridge University Press, 1998.

[4] R. C. Tolman, Phys. Rev. **55**, 364 (1939); J. R. Oppenheimer and G. M. Volkov, Phys. Rev. **55**, 374 (1939).

[5] Steven Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, Inc., New York, 1972, Chapter 11.

[6] M. Prakash, lectures delivered at the Winter School on "The Equation of State of Nuclear Matter," Puri, India, January 1994, esp. Chapter 3, *Equation of State*. These notes are published in "The Nuclear Equation of State", ed. by A. Ausari and L. Satpathy, World Scientific Publishing Co., Singapore, 1996.

[7] R. Balian and J.-P. Blaizot, "Stars and Statistical Physics: A Teaching Experience," Am. J. Phys. **67**, (12) 1189 (1999).

[8] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects*, Wiley-Interscience, 1983.

[9] We apologize to readers who are enthusiasts of SI units, but the first author was raised on CGS units. Actually, we strongly feel that by the time a physics student is at this level, he or she ought to be comfortable in switching from one system of units to another.

[10] A discussion of how to solve these equations (using conventional programming languages) is given in S. Koonin, *Computational Physics*, Benjamin-Cummings Publishing, 1986.

[11] For more details on white dwarfs, NASA provides a useful web page at http://imagine.gsfc.nasa.gov/docs/science/know_l1/dwarfs.html.

[12] This maximum mass of 1.4 $M_\odot$ is usually referred to as the Chandrasekhar limit. See S. Chandrasekhar's 1983 Nobel Prize lecture, http://www.nobel.se/physics/laureates/1983/. For more detail see his treatise, *An Introduction to the Study of Stellar Structure*, Dover Publications, New York, 1939.

[13] Mathematica is a software product of Wolfram Research, Inc., (see web page at http://www.wolfram.com), and its use is described by S. Wolfram in *The Mathematica Book*, Fourth Ed., Cambridge University Press, Cambridge, England, 1999. However, whenever we use the phrase "using Mathematica," we really mean using whatever package one has available or is familiar with, be it Maple, MathCad, or whatever. We did almost all of the numerical/symbolic work that we describe in this paper in Mathematica, but some of its notebooks were duplicated in MathCad, just to be sure it could be done there as well.

[14] Enough of these explicit flags! Most of the equations from here on present challenges for the student to work through.

[15] For the Newtonian case, a polytropic EoS also allows for a somewhat more analytic solution in terms of Lane-Emden functions. See Weinberg, op. cit., Sec. 11.3, or C. Flynn, *Lectures on Stellar Physics*, especially lectures 4 and 5, at http://www.astro.utu.fi/ cflynn/Stars/.

[16] Despite the appearance of the $4\pi\epsilon_0$, the astute student will not be lulled into thinking that this factor has anything to do with a Coulomb potential or the dielectric constant of the vacuum.

[17] Note that the right-hand side of Eq. (22) is negative (for positive $\bar{p}$), so $\bar{p}(r)$ must fall monotonically from $\bar{p}(0)$.

[18] We leave this for the student to figure out, except for the following hint: Use an if statement if necessary.

[19] This fit is least accurate ($\approx 2\%$) at very low values of k_F. However, this is where the pure neutron approximation itself is least accurate. The surface of a neutron star is likely made of elements like iron. A fictional account of what life might be like on such a surface can be found in Robert Forward's *Dragon's Egg*, first published in 1981 by Del Rey Publishing, republished in 2000.

[20] See Weinberg, op. cit., Sec. 11.2.

[21] Because it is almost non-interacting with nuclear matter, a neutrino tends to escape from the neutron star. This is the major cooling mechanism as the neutron star is being formed in a supernova explosion. George Gamow named this the URCA process, after a Brazilian casino where people lost a lot of money.

[22] Does the student know how to put all the factors of c back into ϵ_0 so as to re-write this for CGS units?

[23] See, e.g., J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics*, John Wiley & Sons, 1952, Chap. 6, Sec. 2, or A. Bohr and B.R. Mottelson, *Struktur der Atomkerne*, Akademie-Verlag, Berlin 1975.

[24] The reason for the "(in)" is because a materials physicist might rather define compressibility as $\chi = -(1/V)(\partial V/\partial p) = -(1/n)(dp/dn)^{-1}$.

[25] Folks interested in RHIC physics might want to, however. (RHIC stands for "Relativistic Heavy Ion Collider," an accelerator at the Brookhaven National Laboratory which is studying reactions like Au nuclei striking each other at center of mass energies around 200 GeV/nucleon.)

[26] See, e.g., Hugh Young, *University Physics*, 8th ed., Addison-Wesley, Reading MA, 1992, Sec. 19-5, *Speed of a Longitudinal Wave*.

[27] See, e.g., P. J. E. Peebles and Bharat Ratra, Rev. Mod. Phys. **75**, 599 (2003)

[28] W. Y. Pauchy Hwang, private communication.