Hidden Symmetries in Intrinsic Frame

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What is presented ?

Intrinsic frame

Intrinsic groups

Uniqueness of quantum states

Intrinsic Symmetries of schematic vibration+rotation model

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Tetrahedral nuclei

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Nuclear Tetrahedral Symmetry: Possibly Present throughout the Periodic Table

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More than half a century after the fundamental, spherical shell structure in nuclei had been established, theoretical predictions indicated that the shell gaps comparable or even stronger than those at spherical shapes may exist. Group-theoretical analysis supported by realistic mean-field calculations indicate that the corresponding nuclei are characterized by the T_d^D ("double-tetrahedral") symmetry group. Strong shell-gap structure is enhanced by the existence of the four-dimensional irreducible representations of T_d^D; it can be seen as a *geometrical* effect that does not depend on a particular realization of the mean field. Possibilities of discovering the T_d^D symmetry in experiment are discussed.

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A possibility that atomic nuclei exhibit tetrahedral symmetry — in quantum physics, it is discussed mainly as a property of certain molecules, metal clusters, or fullerines — has a definite interest for all the related domains of physics. While in the above-mentioned objects the underlying interactions are electromagnetic, the nuclear tetrahedra (pyramidlike nuclei with "rounded edges and corners") are expected to be stabilized primarily by the strong in-

glected α_{32} deformation. Using the Hartree-Fock approach in their symmetry-unconstrained variant, Takami *et al.,* Ref. [3], obtain in some light $Z = N$ nuclei an α_{32} instability. In Ref. [4] this and other exotic octupole deformations were studied in the ^{32}S nucleus while, in [5], a similar hypothesis has been advanced theoretically for a group of nuclei around $A \sim 70$.

The experimental verification of the discussed phenome-

Surface collective variables

In the following the surface collective variables will be used as an example of collective variables.

The equation of nuclear surface in the laboratory frame is:

$$R(\theta, \phi) = R_0 \left(1 + \sum_{\lambda\mu} (\alpha^{lab}_{\lambda\mu})^\star Y_{\lambda\mu}(\theta, \phi) \right)$$

$\alpha^{lab}_{\lambda\mu}$ are spherical tensors in respect to $\mathrm{SO}(3)$.

$\mathrm{SO}(3)$ denotes rotation group in the laboratory frame.

Intrinsic frame

An idea of quantum rotating frame

Definition of rotating intrinsic frame for collective variables $\{\alpha_{\lambda\mu}\}$:

1. Let $\{\alpha^{lab}_{\lambda\mu}\}$ = laboratory nuclear collective variables.
and $\{\alpha_{\lambda\mu}\}$ their counterparts (defined below).

2. Let $\mathrm{SO}(3) \ni T(g)$ = rotation group acting in the space $\{\alpha^{lab}_{\lambda\mu}\}$.
The group parameters $g = g(\Omega) = g(\Omega_1, \Omega_2, \Omega_3)$ are intended to be used as a part of intrinsic variables.

3. The transformation formula from the lab. to int. (rotating) frame:

$$\alpha_\lambda = T(g)\alpha^{lab}_\lambda.$$

An idea of quantum rotating frame

The intrinsic variables α_λ are invariant in respect simultaneous action $T(h) \times T_G(h)$, $T_G(h)$ acts on the group manifold of the group $\mathrm{SO}(3)$ by left shift operation

$$T(h) \times T_G(h)\alpha^{lab} = T(h(h^{-1}g))\alpha^{lab} = T(g)\alpha^{lab} = \alpha^{lab}.$$

$T(h) \times T_G(h) =$ a simultaneous rotation of the intrinsic frame and the corresponding laboratory variables by the same angles.

4. Required constraints:

$$F_i(\alpha, \Omega) = 0, \qquad \text{where } i = 1, 2, 3.$$

Classical versus quantum rotation

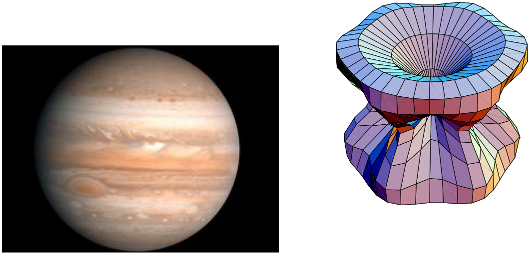

Figure: (left) Motion of a mass as a function of time, (right) The spin orientation probability for a rotating system: $\psi \sim D^5_{M3}(\Omega) - D^5_{M,-3}(\Omega)$.

Intrinsic groups $\overline{\mathrm{G}}$

Jin-Quan Chen, Jialun Ping & Fan Wang: Group Representation Theory for Physicists, World Scientific, 2002.

Def. For each element g of the group G, one can define a corresponding operator $\overline{g}$ in the group linear space $\mathcal{L}_G$ as:

$$\overline{g}S = Sg, \qquad\qquad \text{for all } S \in \mathcal{L}_G.$$

The group formed by the collection of the operators $\overline{g}$ is called the intrinsic group of G.

IMPORTANT PROPERTY:

$$[G, \overline{\mathrm{G}}] = 0$$

The groups G and $\overline{\mathrm{G}}$ are antyisomorphic.

Example: Intrinsic group $\overline{\mathrm{SO}(3)}$

The action of the rotation intrinsic group $\bar{g} \in \overline{\mathrm{SO}(3)}$.

Transformations of coordinates:

$$\alpha^{lab\prime}_{\lambda\mu} = \bar{g}\alpha^{lab}_{\lambda\mu} = \alpha^{lab}_{\lambda\mu}$$

$$\alpha'_{\lambda\mu} = \bar{g}\alpha_{\lambda\mu} = \sum_{\mu'} D_{\mu'\mu}(g^{-1})\alpha_{\lambda\mu'}$$

$$\Omega' = \bar{g}\Omega = \Omega g.$$

The action in the space of functions of intrinsic variables

$$\bar{g}\psi(\alpha, \Omega) = \psi(\bar{g}\alpha, \Omega g^{-1})$$

Uniqueness of quantum states 1/4

In practice, the transformation to intrinsic frame is not a one-to-one function.

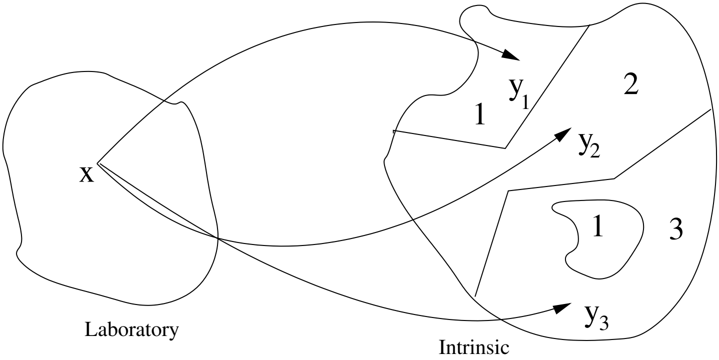

Figure: A one-to-many transformation from laboratory to intrinsic variables.

Uniqueness of quantum states 2/4

To have physical interpretation of some quantities in both laboratory and intrinsic frames one cannot restrict domains of required observables.

Uniqueness achieved by choosing an appropriate subspace of physical states written in intrinsic frame

The construction by making use of a group of transformations $h \in \mathrm{G}_s$:

$$(\alpha, \Omega) \xrightarrow{h} (\alpha', \Omega')$$

which the corresponding laboratory coordinates leave invariant:

$$\alpha^{lab}(\alpha', \Omega') = \alpha^{lab}(\alpha, \Omega)$$

Uniqueness of quantum states 3/4

For an arbitrary function in intrinsic frame usually

$$\Psi(\alpha, \Omega) \neq \Psi(\alpha', \Omega'),$$

but

$$\Psi(\alpha, \Omega) = \Psi(\alpha^{lab})$$
$$\Psi(\alpha', \Omega') = \Psi(\alpha^{lab}).$$

CONTRADICTION.

The group $\overline{\mathrm{G}}_s$ is called the SYMMETRIZATION GROUP.

Uniqueness of quantum states 4/4

REQUIRED !

The symmetrization condition for states. For all $\bar{h} \in \overline{\mathrm{G}}_s$:

$$\bar{h}\Psi(\alpha, \Omega) = \Psi(\alpha, \Omega)$$

Example: Symmetrization group

Let us consider the **standard** choice of collective quadrupole variables $(\alpha_{20}, \alpha_{22}, \Omega)$:

$$F_{1,2}(\alpha, \Omega) = \alpha_{2\pm1} = 0 \text{ and } F_3(\alpha, \Omega) = \alpha_{2-2} - \alpha_{22} = 0$$
$$\Rightarrow \overline{\mathrm{G}}_s = \overline{\mathrm{O}}_h$$

Another choice of intrinsic variables $(\alpha_{20}, \alpha_{21}, \Omega)$:

$$F_{1,2}(\alpha, \Omega) = \alpha_{2\pm2} = 0 \text{ and } F_3(\alpha, \Omega) = \alpha_{21} + \alpha_{2-1} = 0$$
$$\Rightarrow \overline{\mathrm{G}}_s = \overline{\mathrm{D}}_{2h}$$

Symmetry operations in the intrinsic frame

The operations allowed in the intrinsic variables space:

All operations which fulfil the conditions:

$$F_i(\{\alpha_{\lambda\mu}\}, \Omega) = 0, \;\; i = 1, 2, 3.$$

save the structure of intrinsic variables space.

EXAMPLES:

- Ex.1.: $\overline{\mathrm{O}(3)}^{(rot)}$ which acts only on the rotational degrees of fredom.
- Ex.2.: $\overline{\mathrm{SO}(3)}^{(quad)}$ acting only on quadrupole variables:

$$\alpha'_{2\mu} = \sum_{\mu'} D^2_{\mu'\mu}(\xi)\alpha_{2\mu'}$$

fulfil the required conditions.

Intrinsic Symmetries of schematic vibration+rotation model

Hamiltonian = oscillator + rotor

The variables ($q_1 = \sqrt{2}\alpha_{22}, q_2 = \alpha_{20}, \Omega$) with standard constraints.

The quadrupole+rotor model Hamiltonian:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{vib} + \hat{\mathcal{H}}_{rot},$$

where

$$\hat{\mathcal{H}}_{vib} = -\frac{\hbar^2}{2B}\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2}\right) + \frac{1}{2}B\omega^2(q_1^2 + q_2^2).$$

$$\hat{\mathcal{H}}_{rot} = \hat{\mathcal{H}}_{rot}(J_x, J_y, J_z)$$

and $[\hat{\mathcal{H}}_{rot}, \overline{\mathrm{SO}(3)}^r] = 0$
No vib-rot coupling terms $\Rightarrow$ **the eigenfunctions:**

$$\Psi_{\Gamma a; JM\nu}(\alpha, \Omega) = \phi_{\Gamma a}(\alpha) R_{JM\nu}(\Omega)$$

Symmetry group chain

$$\begin{array}{ccccccc} & & \hat{\mathcal{H}} & = & \hat{\mathcal{H}}_{vib} & + & \hat{\mathcal{H}}_{rot} \\ & & \downarrow & & \downarrow & & \downarrow \\ \overline{\mathrm{G}}_s & \subset & \overline{\mathrm{G}}_H & = & \overline{\mathrm{G}}_{vib} & \times & \overline{\mathrm{G}}_{rot} \\ \downarrow & \downarrow & & & \downarrow & & \downarrow \\ \Gamma_s = 0 & \sigma & & & \Gamma_v & & \Gamma_r \end{array}$$

The symmetrization group G_s is NOT a PHYSICAL SYMMETRY group of $\hat{\mathcal{H}}$.

The physical symmetry of $\hat{\mathcal{H}}$ should be constructed from the formal symmetry G_H after "subtracting" of the symmetrization group G_s.

Transitions

The $\mathrm{SO}(3)$–reduced matrix elements of the multipole transition operator:

$$\langle \Psi_{\Gamma' a'; J'\nu'} || Q_\lambda^{lab} || \Psi_{\Gamma a; J\nu} \rangle = \sum_\mu \langle \phi_{\Gamma' a'} | Q_{\lambda\mu} | \phi_{\Gamma a} \rangle \langle R_{J'\nu'} || D_{\cdot\mu}^{\lambda\star} || R_{J\nu} \rangle$$

The reduced probability:

$$B(E\lambda; (\Gamma a; J\nu) \to (\Gamma' a'; J'\nu')) = |\langle \Gamma' a'; J'\nu' || Q_\lambda^{lab} || \Gamma a; J\nu \rangle|^2 / (2J+1)$$

Partial symmetries of $\hat{\mathcal{H}}$ can be responsible for some selection rules !
Still, an OPEN PROBLEM

Vibrational Hamiltonian

Boson creation-anihilation operators

$$b_k^+ = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{B\omega}{\hbar}} - i\sqrt{\frac{1}{B\hbar\omega}}\left(-i\frac{\partial}{\partial q_k}\right)\right]$$
$$b_k = (b_k^+)^\dagger$$

The vibrational Hamiltonian can be rewritten:

$$\hat{\mathcal{H}}_{vib} = \hbar\omega(\hat{N} + 1),$$

where $\hat{N} = b_1^+ b_1 + b_2^+ b_2$.

The formal symmetry group $\overline{\mathrm{G}}_{vib} = \overline{\mathrm{U}(2)}^{(vib)}$

Vibrational $\overline{\mathrm{U}(2)}^{(vib)}$

The 2-dim i.r. $g_v(\vartheta, a, b) \in \overline{\mathrm{U}(2)}^{(vib)}$ acts only on vibrational degrees of freedom:

$$g_v(\vartheta, a, b) b_k^+ g_v(\vartheta, a, b)^{-1} = \sum_{k'=1}^{2} \Delta_{k'k}^{(U,\frac{1}{2})}(\vartheta, a, b) b_{k'}^+$$

$$g_v(\vartheta, a, b)\Psi(\Omega) = \Psi(\Omega),$$

where

$$\Delta^{(U,\frac{1}{2})}(\vartheta, a, b) = \exp(i\vartheta) \begin{pmatrix} a & b \\ -b^\star & a^\star \end{pmatrix},$$

where $|a|^2 + |b|^2 = 1$.

The vibrational octahedral group $\overline{\mathrm{O}}^{(vib)} \subset \overline{\mathrm{U}(2)}^{(vib)}$

Rotational $\overline{\mathrm{SO}(3)}^{(rot)}$

$g_R(\theta = (\theta_1, \theta_2, \theta_3)) \in \overline{\mathrm{SO}(3)}^{(rot)}$ acts only on rotational degrees of freedom:

$$g_R(\theta) b_k^+ g_R(\theta)^{-1} = b_k^+$$
$$g_R(\theta)\Psi(\Omega) = \Psi(\Omega\theta^{-1}).$$

The rotational octahedral group $\overline{\mathrm{O}}^{(rot)} \subset \overline{\mathrm{SO}(3)}^{(rot)}$.

NOTE:

The symmetrization group $\overline{\mathrm{G}}_s = \overline{\mathrm{O}} \subset \overline{\mathrm{O}}^{(vib)} \times \overline{\mathrm{O}}^{(rot)}$

Hidden intrinsic symmetries of $\hat{\mathcal{H}}$

One needs to find the symmetry operations $h \in \overline{\mathrm{G}}_H$ (Hamiltonian formal symmetry group) which do not belong to G_s (symmetrization group).

Consider the single-phonon function:

$$|N = 1, JM\rangle = \sqrt{\frac{5}{2}} \left(\frac{1}{\sqrt{2}} b_1^+ (D^2_{M2}(\Omega)^\star + D^2_{M,-2}(\Omega)^\star) + b_2^+ D^2_{M0}(\Omega)^\star \right) |0\rangle.$$

This state is G_s invariant.

Hidden intrinsic symmetries of $\hat{\mathcal{H}}$

The additional operations $h \in \overline{\mathrm{G}}_H$ and $h \notin \overline{\mathrm{G}}_s$ which leave the vector $|N = 1, JM\rangle$ in the same 1-Dim subspace are

$$(\exp(i\vartheta)e_G, \bar{C}_{2l}), \text{ where } l = x, y, z.$$

Hidden symmetries of $\hat{\mathcal{H}}$, $N = 1$ and $J = 2$

$\overline{\mathrm{G}}_{phys} = U(1)^{(vib)} \times \overline{\mathrm{D}}_2^{(rot)} \subset \overline{\mathrm{G}}_H$

Hidden symmetries of $\hat{\mathcal{H}}$ for other N and J

$$\overline{\mathrm{G}}_{phys} = ?????$$

OPEN PROBLEM !!!

A few conclusions

- One can choose different kinds of rotating intrinsic frames.
- Usually the transformation to intrinsic frame is not unique. Because of physical reasons, instead of cutting domains of intrinsic variables, one needs to introduce the symmetrization group $\overline{\mathrm{G}}_s$.
- The physical space consists of $\overline{\mathrm{G}}_s$-scalar functions of intrinsic variables.
- The physical symmetries of the intrinsic Hamiltonian are hidden and should be obtained after subtraction the symmetrization group $\overline{\mathrm{G}}_s$ from the formal symmetry group $\overline{\mathrm{G}}_H$.

SUMMARY

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