XIII Nuclear Physics Workshop

Marie and Pierre Curie

Pairing & Beyond – 50 Years of the BCS Model

Kazimierz Dolny, Poland, September 27-October 1, 2006

Exactly solvable pairing Hamiltonians

50 of Cooper pairs

49 years of the BCS model

43 years of the exact solution

1) The Cooper problem (1956)
2) BCS approximation (1957)
3) BCS in nuclear structure (1958)
4) Richardson exact solution (1963)
5) Gaudin magnet (1976)
6) Rank 1 Richardson-Gaudin integrable models (2001)
7) Higher rank RG integrable models. T=1 pairing (2006).
8) SO(8) T=0,1 pairing model. A rank 4 RG integrable model.

The Cooper Problem

PHYSICAL REVIEW VOLUME 104, NUMBER 4 NOVEMBER 15, 1956

Letters to the Editor

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Bound Electron Pairs in a Degenerate Fermi Gas*

LEON N. COOPER
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(Received September 21, 1956)

IT has been proposed that a metal would display superconducting properties at low temperatures if the one-electron energy spectrum had a volume inde-

$=(1/V)\exp[i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)]$ which satisfy periodic boundary conditions in a box of volume V, and where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of electron one and electron two. (One can use antisymmetric functions and obtain essentially the same results, but alternatively we can choose the electrons of opposite spin.) Defining relative and center-of-mass coordinates, $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, $\mathbf{r}=(\mathbf{r}_2-\mathbf{r}_1)$, $\mathbf{K}=(\mathbf{k}_1+\mathbf{k}_2)$ and $\mathbf{k}=\frac{1}{2}(\mathbf{k}_2-\mathbf{k}_1)$, and letting $\mathcal{E}_K+\epsilon_k=(\hbar^2/m)(\frac{1}{4}K^2+k^2)$, the Schrödinger equation can be written

$$(\mathcal{E}_K+\epsilon_k-E)a_\mathbf{k}+\sum_{\mathbf{k}'}a_{\mathbf{k}'}(\mathbf{k}|H_1|\mathbf{k}')\times\delta(\mathbf{K}-\mathbf{K}')/\delta(0)=0 \quad (1)$$

where

$$\Psi(\mathbf{R},\mathbf{r})=(1/\sqrt{V})e^{i\mathbf{K}\cdot\mathbf{R}}\chi(\mathbf{r},K),\quad \chi(\mathbf{r},K)=\sum_\mathbf{k}(a_\mathbf{k}/\sqrt{V})e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (2)$$

and

$$(\mathbf{k}|H_1|\mathbf{k}')=\left(\frac{1}{V}\int d\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}H_1e^{i\mathbf{k}'\cdot\mathbf{r}}\right)_{0\text{ phonons}}.$$

Problem : A pair of electrons with an attractive interaction on top of an inert Fermi sea.

$$|\phi\rangle=\sum_{k>k_F}\frac{1}{2\varepsilon_k-E}c^+_{k\uparrow}c^+_{-k\downarrow}|FS\rangle,\qquad \frac{1}{G}=\sum_{k>k_F}\frac{1}{2\varepsilon_k-E}$$

Bound pair for arbitrary small attractive interaction. The FS is unstable against pair formation. Predicts a gap.

If the many-body system could be considered (at least to a lowest approximation) a collection of pairs of this kind above a Fermi sea, we would have (whether or not the pairs had significant Bose properties) a model similar to that proposed by Bardeen which would display many of the equilibrium properties of the superconducting state.

Bardeen-Cooper-Schrieffer

PHYSICAL REVIEW VOLUME 108, NUMBER 5 DECEMBER 1, 1957

Theory of Superconductivity*

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(Received July 8, 1957)

A theory of superconductivity is presented, based on the fact that the interaction between electrons resulting from virtual exchange of phonons is attractive when the energy difference between the electrons states involved is less than the phonon energy, $\hbar\omega$. It is favorable to form a superconducting phase when this attractive interaction dominates the repulsive screened Coulomb interaction. The normal phase is described by the Bloch individual-particle model. The ground state of a superconductor, formed from a linear combination of normal state configurations in which electrons are virtually excited in pairs of opposite spin and momentum, is lower in energy than the normal state by amount proportional to an average $(\hbar\omega)^2$, consistent with the isotope effect. A mutually orthogonal set of excited states in one-to-one correspondence with those of the normal phase is obtained by specifying occupation of certain Bloch states and by using the rest to form a linear combination of virtual pair configurations. The theory yields a second-order phase transition and a Meissner effect in the form suggested by Pippard. Calculated values of specific heats and penetration depths and their temperature variation are in good agreement with experiment. There is an energy gap for individual-particle excitations which decreases from about $3.5kT_c$ at $T=0$°K to zero at T_c. Tables of matrix elements of single-particle operators between the excited-state superconducting wave functions, useful for perturbation expansions and calculations of transition probabilities, are given.

For simplicity, consider the reduced BCS Hamiltonian

$$H_P = \sum_k \varepsilon_k n_k + \frac{G}{V}\sum_{k,k'} c^{\dagger}{}_{k\uparrow} c_{-k\downarrow}{}^{\dagger} c_{k'\downarrow} c_{k'\uparrow}$$

$$\left|\Psi\right\rangle \equiv e^{\Gamma^+}\left|0\right\rangle, \quad \Gamma^+ = \sum_k \frac{v_k}{u_k} c^+_{k\uparrow} c^+_{-k\downarrow}$$

BCS in Nuclear Structure

PHYSICAL REVIEW VOLUME 110, NUMBER 4 MAY 15, 1958

Possible Analogy between the Excitation Spectra of Nuclei and Those of the Superconducting Metallic State

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(Received January 7, 1958)

The evidence for an energy gap in the intrinsic excitation spectrum of nuclei is reviewed. A possible analogy between this effect and the energy gap observed in the electronic excitation of a superconducting metal is suggested.

It thus appears that there may exist interesting similarities between the low-energy spectra of nuclei and of the electrons in the superconducting metal. However, it must be stressed that the former are significantly influenced by the finite size of the nuclear system. Thus, the energy gap is observed to decrease

Number Projected BCS

PHYSICAL REVIEW VOLUME 135, NUMBER 1B 13 JULY 1964

Conservation of Particle Number in the Nuclear Pairing Model*

KLAUS DIETRICH, HANS J. MANG, AND JEAN H. PRADAL
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(Received 13 January 1964; revised manuscript received 26 February 1964)

The Euler–Lagrange equations corresponding to a Bardeen–Cooper–Schrieffer state that is an eigenstate of the number operator are derived and solved numerically for a δ interaction. The errors due to the non-conservation of particle number in the usual Bardeen–Cooper–Schrieffer theory are studied as a function of particle number, level density, and strength of the pairing interaction. A proof is given that for attractive pairing interactions the lowest energy solution corresponds always to real positive probability amplitudes v_ν, u_ν.

$$\left|\Psi\right\rangle \equiv \left(\Gamma^{+}\right)^{M}\left|0\right\rangle, \quad \Gamma^{+} = \sum_{k} \alpha_k c_{k\uparrow}^{+} c_{-k\downarrow}^{+}$$

Richardson’s Exact Solution

ume 3, number 6 PHYSICS LETTERS 1 February 1963

A RESTRICTED CLASS OF EXACT EIGENSTATES OF THE PAIRING-FORCE HAMILTONIAN *

R. W. RICHARDSON
H. M. Randall Laboratory of Physics,
University of Michigan, Ann Arbor, Michigan

Received 23 November 1962

Exact Solution of the BCS Model

R.W. Richardson: Phys. Lett. **3**, 277 (1963); Phys. Rev. **141**, 949 (1966).

$$H_P = \sum_k \varepsilon_k n_k + g \sum_{k,k'} c^\dagger{}_{k\uparrow} c_{-k\downarrow}{}^\dagger c_{k'\downarrow} c_{k'\uparrow}$$

Eigenvalue equation:

$$H_P \left|\Psi\right\rangle = E \left|\Psi\right\rangle$$

Ansatz for the eigenstates (generalized Cooper ansatz)

$$\left|\Psi\right\rangle = \prod_{\alpha=1}^{M} \Gamma_\alpha^\dagger \left|\nu\right\rangle, \quad \Gamma_\alpha^\dagger = \sum_k \frac{1}{2\varepsilon_k - E_\alpha} c_{k\uparrow}^+ c_{-k\downarrow}^+$$

$$c_{k\sigma}^+ c_{k\sigma} \left|\nu\right\rangle = \nu_{k\sigma} \left|\nu\right\rangle, \quad c_{-k\downarrow} c_{k\uparrow} \left|\nu\right\rangle = 0$$

Richardson equations

$$1+g\sum_{k=0}\frac{1-\nu_k}{2\varepsilon_k-E_\alpha}+2g\sum_{\beta(\neq\alpha)=1}^{M}\frac{1}{E_\alpha-E_\beta}=0,\;\; E=\sum_{\alpha=1}^{M}E_\alpha$$

Properties:

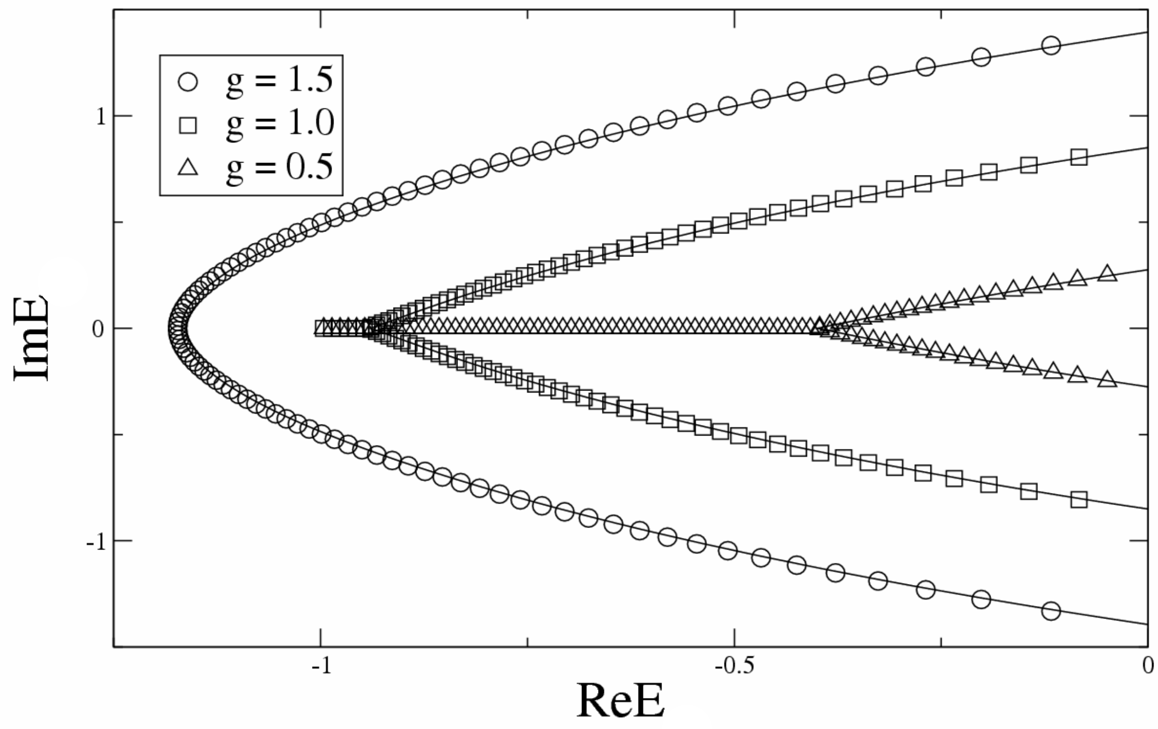
This is a set of *M* nonlinear coupled equations with *M* unknowns (E_α).

The first and second terms correspond to the equations for the one pair system. The third term contains the many body correlations and the exchange symmetry.

The pair energies are either real or complex conjugated pairs. Complex pair energies imply correlated pairs. Cooper pair?

There are as many solutions as states in the Hilbert space. The solutions can be classified in the weak coupling limit ($g\rightarrow 0$).

Pair energies E for a system of 200 equidistant levels at half filling

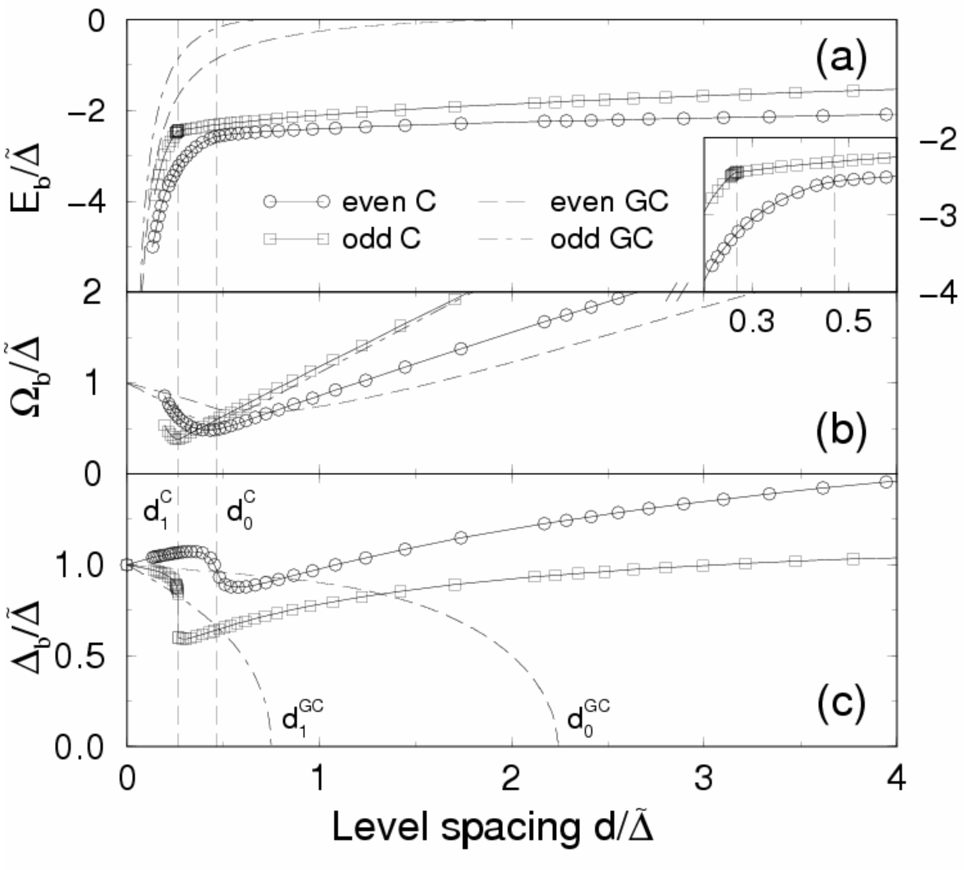


Recovery of the Richardson solution: Ultrasmall superconducting grains

- A fundamental questions posed by P.W. Anderson in J. Phys. Chem. Solids 11 (1959) 26 :
- "at what size of particles will superconductivity actually cease?"
- Anderson argued that for a sufficiently small metallic particle, since $d \sim Vol^{-1}$, there will be a critical size $d \sim \Delta_{bulk}$ at which superconductivity must disappear.
- This condition indeed arises for grains at the nanometer scale.
- Main motivation from the revival of this old question came from the works:
- D.C. Ralph, C. T. Black y M. Tinkham
- PRL's 74 (1995) 3421 ; 76 (1996) 688 ; 78 (1997) 4087.

PBCS study of ultrasmall grains:

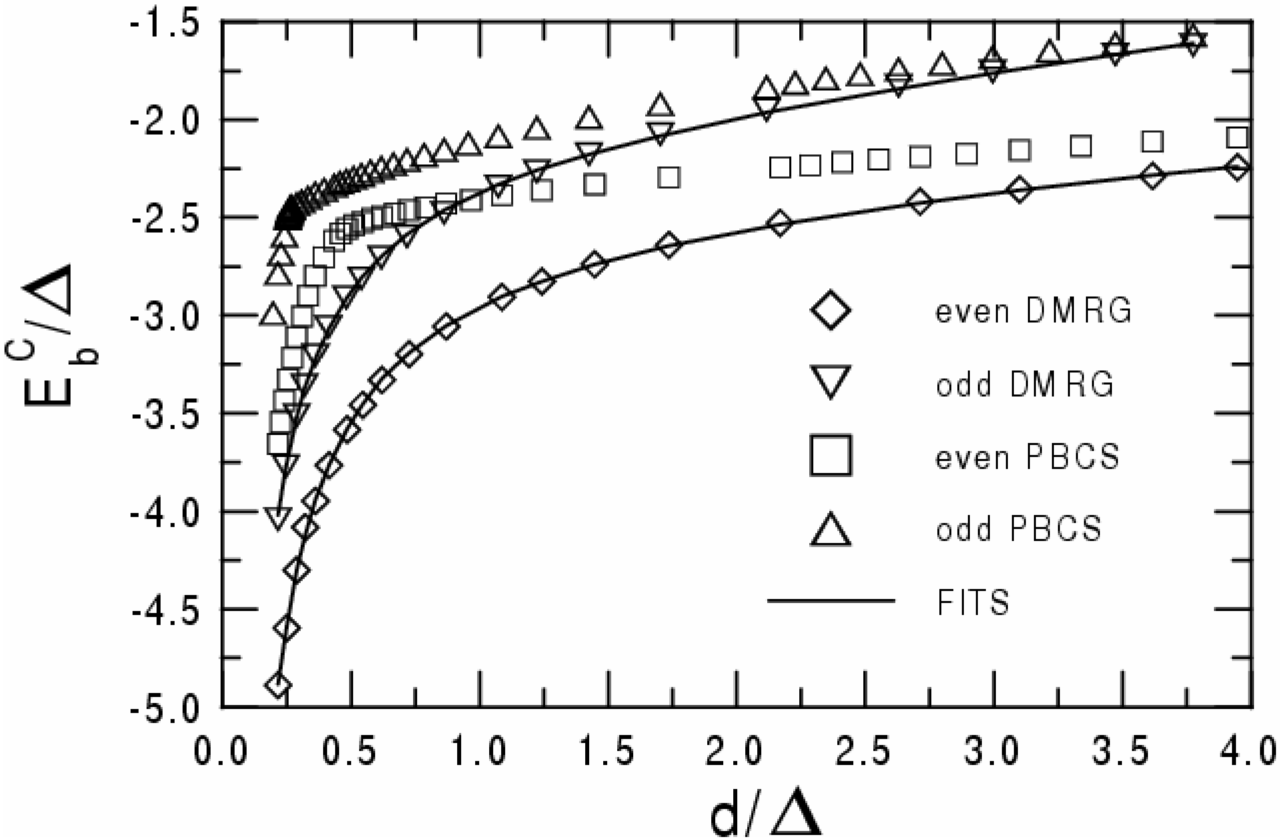
•Braun y J. von Delft. PRL **81** (1998)47



Condensation energy for even and odd grains

PBCS versus Exact

JD and G. Sierra, PRL 83, 172 (1999)



Richardson-Gaudin Models

JD, C. Esebbag and P. Schuck, PRL 87, 066403 (2001).

- Combine the Richardson's exact solution of the Pairing Model and the integrable Gaudin Magnet

- Based on the rank 1 pair algebra of su(2).

$$\left[J^0, J^{\pm}\right] = \pm J^{\pm}, \quad \left[J^+, J^-\right] = \mp 2J^0$$

1) Pair realization

$$J_k^0 = \frac{1}{2}\left(a_k^+ a_k + a_{\bar{k}}^+ a_{\bar{k}} \pm 1\right), \; J_k^+ = a_k^+ a_{\bar{k}}^+$$

2) Two-level realization

$$J_k^0 = \frac{1}{2}\sum_m (a_{k+m}^+ a_{k+m} - a_{k-m}^+ a_{k-m}), \; J_k^+ = \sum_m a_{k+m}^+ a_{k-m}$$

3) Finite center of mass momentum realization (LOFF)

$$J_{k,Q}^0 = \frac{1}{2}\left(a_{k+Q}^+ a_{k+Q} + b_{-k}^+ b_{-k} - 1\right), \quad J_{k,Q}^+ = a_{k+Q}^+ b_{-k}^+$$

4) Spin or Angular momentun realization

Construction of the Integrals of Motion

- The most general combination of linear and quadratic generators, with the restriction of being hermitian and number conserving, is

$$R_i = J_i^0 + 2g \sum_{j(\neq i)} \left[\frac{X_{ij}}{2} \left(J_i^+ J_j^- + J_i^- J_j^+ \right) + Y_{ij} J_i^0 J_j^0 \right]$$

- The integrability condition $\left[R_i, R_j \right] = 0$ leads to

$$Y_{ij} X_{jk} + X_{jk} Y_{ki} + X_{ki} X_{ij} = 0$$

- These are the same conditions encountered by Gaudin (J. de Phys. 37 (1976) 1087) in a spin model known as the Gaudin magnet.

•Gaudin (1976) found three solutions

- Rational Model

$$X_{ij} = Y_{ij} = \frac{1}{\eta_i - \eta_j}$$

- Exact Solution

$$R_i \left| \Psi \right\rangle = r_i \left| \Psi \right\rangle$$

- Richardson equations

$$1 + g \sum_j \frac{\Omega_j}{2\eta_j - e_\alpha} + 4g \sum_{\beta(\neq\alpha)} \frac{1}{e_\alpha - e_\beta} = 0$$

- Eigenvalues

$$r_i = -\frac{\Omega_i}{4} \left[1 - \frac{1}{2} \sum_{j(\neq i)} \frac{1}{\eta_i - \eta_j} + 4g \sum_\alpha \frac{1}{2\eta_i - e_\alpha} \right]$$

Any function of the *R* operators defines a valid integrable hamiltonian. The hamiltonian is diagonal in the basis of common eigenstates of the *R* 0perators.

•Two body Hamiltonians can be obtain by a linear combination of *R* operators:

$$H = \sum_l \varepsilon_l \, R_l\left(\eta, g\right) + C$$

•The parameters *g*, η´*s* and ε´*s* are arbitrary. There are 2 *L*+1 free parameters to define an integrable hamiltonian in each of the three models (*L* being the number of single particle levels).

•The rational model for fermions is precisely the integrable model proposed by CRS (NPA 1997). They showed that the PM hamiltonian can be obtained from (3) by choosing $\eta = \varepsilon$.

•An important difference between RG models and other ES models is the large number of free parameters.

Some models derived from RG

BCS Hamiltonian (Fermion and Boson)

Generalized Pairing Hamiltonians (Fermion and Bosons)

Central Spin Model

Spin models

Lipkin Model

Multilevel-level boson models (Trapped bosons, IBM, molecular, etc..)

Atom-molecule Hamiltonians (Feshbach resonances)

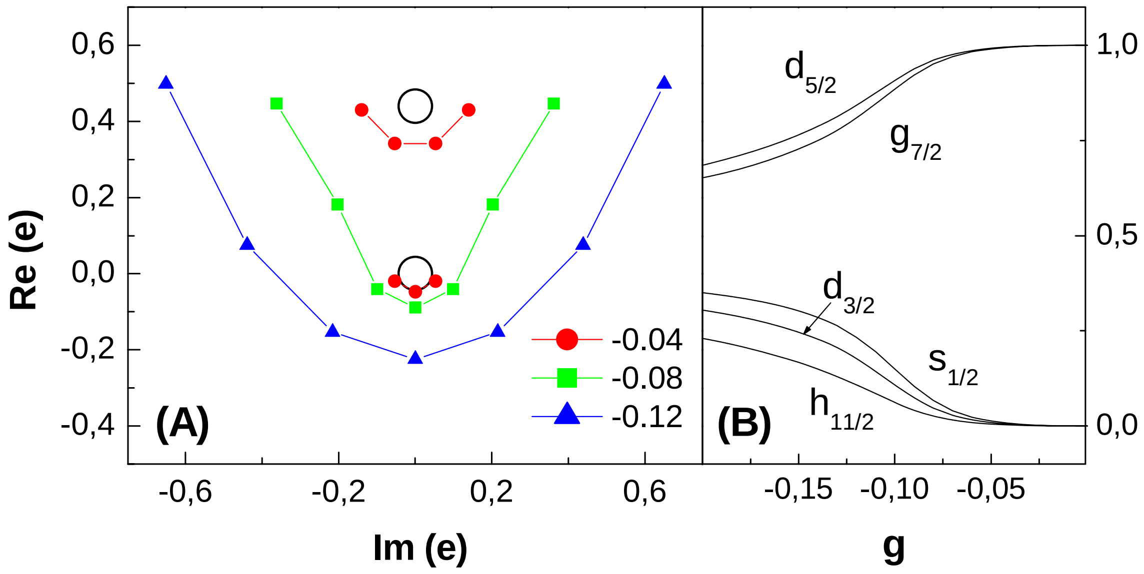
Generalized Tavis-Cummins models.

LOFF Hamiltonians

Review article: JD, S. Pittel and G. Sierra, Rev. Mod. Phys. 76 (2004) 643.

^{114}Sn

J. D, C. Esebbag y S. Pittel. PRL 88 (2002) 062501.



Generalized Richardson-Gaudin Models

- RG Models are Exactly Solvable for any simple Lie algebra
- Rank 2 includes O(5), O(3,2), SU(3)
- O(5) Isovector pairing model, High Tc superconductivity.

  R.W. Richardson, Phys. Rev. 144, 874 (1966).

  J. Links *et al. J.* Phys. A 35, 6459 (2002).

  F. Pan and J. Draayer, Phys. Rev. C 66, 044314 (2002).

  J. D. *et al.* Phys. Rev. Lett. 96, 072503 (2006).
- O(3,2) IBM2, two-species Bose condensates. S. Lerma *et al.* PRC 74 (2006) 024314.
- SU(3) Three level atoms, Generalized Tavis-Cummings models, etc. In preparation. Elliot?
- Rank 4 SO(8) model. Work in progress

The SO(5) algebra

For each copy of Lie Algebra (associated with a single-particle level), the following 10 generators satisfy the SO(5) commutation relationships:

- $SU_T(2)$ subalgebra:

$$T_i^0 = \frac{1}{2}(p_i^+ p_i + p_{\bar{i}}^+ p_{\bar{i}}) - \frac{1}{2}(n_i^+ n_i + n_{\bar{i}}^+ n_{\bar{i}}) \qquad T_i^+ = \frac{1}{\sqrt{2}}(p_i^+ n_i + p_{\bar{i}}^+ n_{\bar{i}}) \qquad T_i^- = \frac{1}{\sqrt{2}}(n_i^+ p_i + n_{\bar{i}}^+ p_{\bar{i}})$$

- T=1 Pair creation operators:

$$b_{-1i}^+ = n_i^+ n_{\bar{i}}^+ \qquad b_{0i}^+ = \frac{1}{\sqrt{2}}(n_i^+ p_{\bar{i}}^+ + p_i^+ n_{\bar{i}}^+) \qquad b_{+1i}^+ = p_i^+ p_{\bar{i}}^+$$

- Second Cartan operator

$$H_i = \frac{1}{2} \sum_{\rho = p,n} (\rho_i^+ \rho_i + \rho_{\bar{i}}^+ \rho_{\bar{i}}) - 1$$

Integrals of motion and eigenvalues

$$R_i = (2+\Delta)H_i + \Delta T_i^0 + 2g \sum_{i'(\neq i)=1}^{L} \frac{1}{z_i - z_{i'}} [\sum_{\mu} (b_{\mu i}^{+} b_{\mu i'} + b_{\mu i'}^{+} b_{\mu i}) + T_i \cdot T_{i'} + H_i H_{i'}]$$

$$r_i = 2g \sum_{i'(\neq i)=1}^{L} \frac{\left(\frac{\nu_i}{2}-1\right)\left(\frac{\nu_{i'}}{2}-1\right) + t_i t_{i'}}{z_i - z_{i'}} - g \sum_{\beta=1}^{M+T_0+t} \frac{2t_i}{z_i - \omega_\beta} + g \sum_{\alpha=1}^{M} \frac{\nu_i + 2t_i - 2}{z_i - e_\alpha} + \left[(2+\Delta)\left(\frac{\nu_i}{2}-1\right) - \Delta t_i\right]$$

Richardson equations

$$\frac{1}{g} = \frac{1}{2} \sum_{i=1}^{L} \frac{\nu_i + 2t_i - 2}{z_i - e_\alpha} + \sum_{\alpha'(\neq \alpha)=1}^{M} \frac{2}{e_{\alpha'} - e_\alpha} - \sum_{\beta=1}^{M+T_0+t} \frac{1}{\omega_\beta - e_\alpha},$$

$$\frac{\Delta}{g} = \sum_{\beta'(\neq \beta)=1}^{M+T_0+t} \frac{2}{\omega_{\beta'} - \omega_\beta} - \sum_{\alpha=1}^{M} \frac{2}{e_\alpha - \omega_\beta} - \sum_{i=1}^{L} \frac{2t_i}{z_i - \omega_\beta}.$$

Neutron-proton pairing Hamiltonian

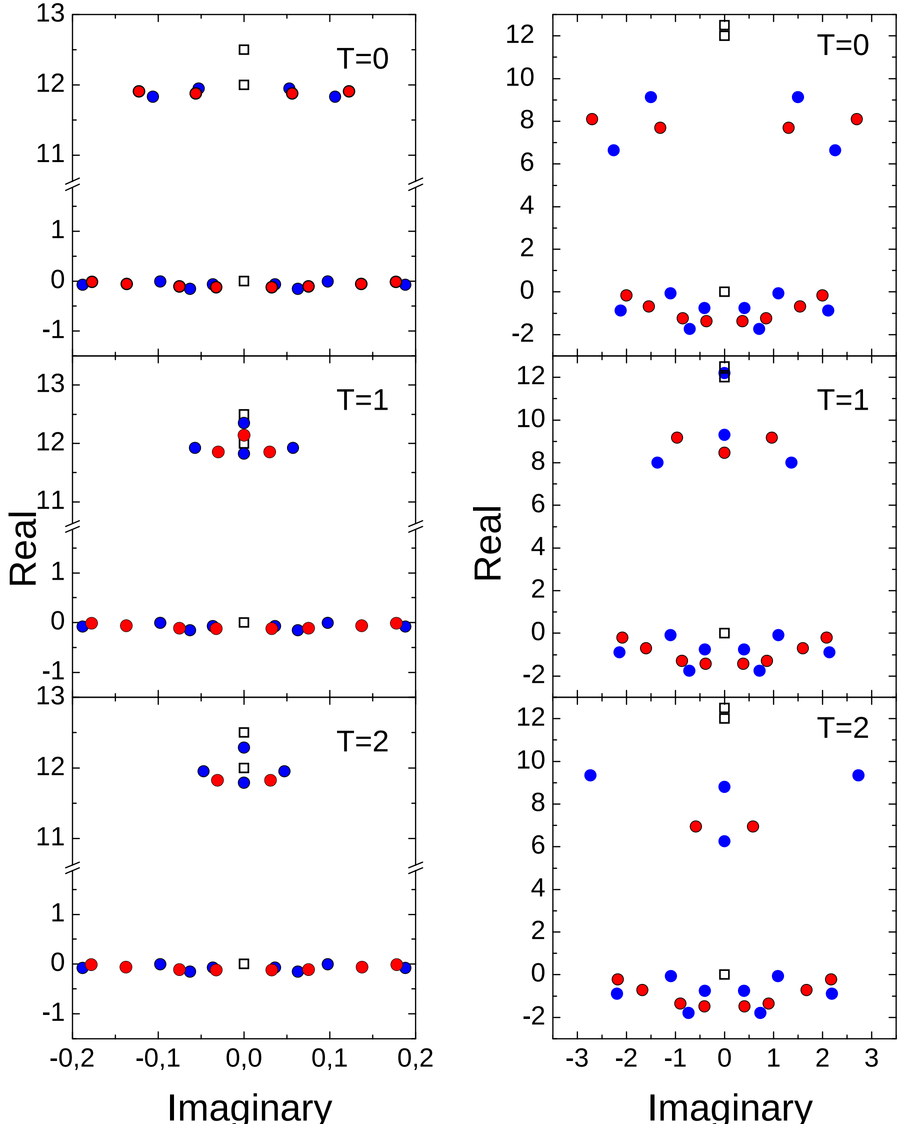
$$\hat{H} = \frac{1}{2}\sum_{i=1}^{L} z_i \hat{R}_i + g\hat{C}$$

$$= \sum_j \varepsilon_j(\hat{N}_j + \Delta\hat{N}_{p,j}) + \frac{g}{2}\hat{T}\cdot\hat{T} + g\sum_{\mu\, jmj'm'} \hat{b}^{\dagger}_{\mu,jm}\hat{b}_{\mu,j'm'},$$

$$E = \sum_{\alpha=1}^{M} e_\alpha + \frac{\Delta}{2}\sum_{\beta=1}^{M+T_0+t}\omega_\beta + \sum_i \varepsilon_i\left[\frac{\nu_i}{2}(2+\Delta) - \Delta t_i\right] + \frac{g}{2}T_0(T_0-1)$$

^{64}Ge in the pfg$_{9/2}$ valence space

JD et al. PRL 96 (2006) 072503



$$2\varepsilon_{f_{7/2}} = 0.0, \quad 2\varepsilon_{p_{3/2}} = 6.00,$$

$$2\varepsilon_{f_{5/2}} = 6.25, \quad 2\varepsilon_{p_{1/2}} = 7.10,$$

$$2\varepsilon_{g_{9/2}} = 9.6$$

$\{\omega\}$ and {e} for the weak and strong coupling limit for T=0,1,2 states

The SO(8) RG model

Pair operators:

$$P^{+}_{l\tau} = \sqrt{\frac{2l+1}{2}}\left[c^{+}_{l}c^{+}_{l}\right]^{001}_{00\tau}, \quad D^{+}_{l\tau} = \sqrt{\frac{2l+1}{2}}\left[c^{+}_{l}c^{+}_{l}\right]^{010}_{0\sigma 0}$$

Exactly solvable SO(8) Hamiltonian

$$H = \sum_{l} \varepsilon_{l}\, n_{l} + g \sum_{l,l'} \left[\sum_{\tau} P^{+}_{l\tau} P_{l'\tau} + \sum_{\tau} D^{+}_{l\sigma} D_{l'\sigma} \right]$$

SUMMARY

rank	A_n *su*(n+1)	B_n *so*(2n+1)	C_n *sp*(2n)	D_n *so*(2n)
1	***su*(2) pairing**	*so*(3)~*su*(2)	***sp*(2) ~*su*(2)**	*so*(2) ~*u*(1)
2	*su*(3) Elliott?	***so*(5) pn-pairing**	***sp*(4) ~*so*(5)**	*so*(4) ~*su*(2)x*su*(2)
3	*su*(4) Wigner	*so(7)* *FDSM*	*sp*(6) Symplectic, FDSM	*so*(6)~*su*(4)
4	*su*(5)	*so(9)*	*sp*(8)	*so*(8) T=0,1 pairing; FDSM

Closing Remarks

- Overview of the exactly-solvable Richardson-Gaudin models.
- There has been major recent progress within the SU(2) and SU(1,1) models. Applications to nuclear physics and cold atomic gases.
- Extension of the RG models to simple groups. Rank 2: SO(5), SO(3,2), SU(3). Rank 4 SO(8).
- We can anticipate new interesting developments in the future.