Kazimierz Dolny, 28 Sept., 2006

Renormalization Group approach to the pairing instabilities

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http://kft.umcs.lublin.pl/doman/lectures

Outline

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- **Introduction**

/ pairing in the many-body systems /

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- **New RG formulation**

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- **Applications**

  / within and beyond the BCS framework /

I. Introduction

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Very often formation of the fermion pairs goes hand in hand with **superconductivity/superfluidity** but it needs not be the rule.

Hamiltonian of the pairing interactions

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The momentum representation:

$$\hat{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \; \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \, \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \, \hat{c}_{-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{k}'\uparrow}$$

where $V_{\mathbf{k},\mathbf{k}'} < 0$ (at least for some $\mathbf{k}$, $\mathbf{k}'$ states)

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The real space representation:

$$\hat{H} = \sum_{i,j} \sum_{\sigma} t_{i,j} \hat{c}^{\dagger}_{i\sigma} \hat{c}_{j\sigma} + \sum_{i,j} V_{i,j} \; \hat{c}^{\dagger}_{i\uparrow} \; \hat{c}_{i\uparrow} \; \hat{c}^{\dagger}_{j\downarrow} \hat{c}_{j\downarrow}$$

with attractive potential $V_{i,j} < 0$

II. Renormalization Group

Strategy

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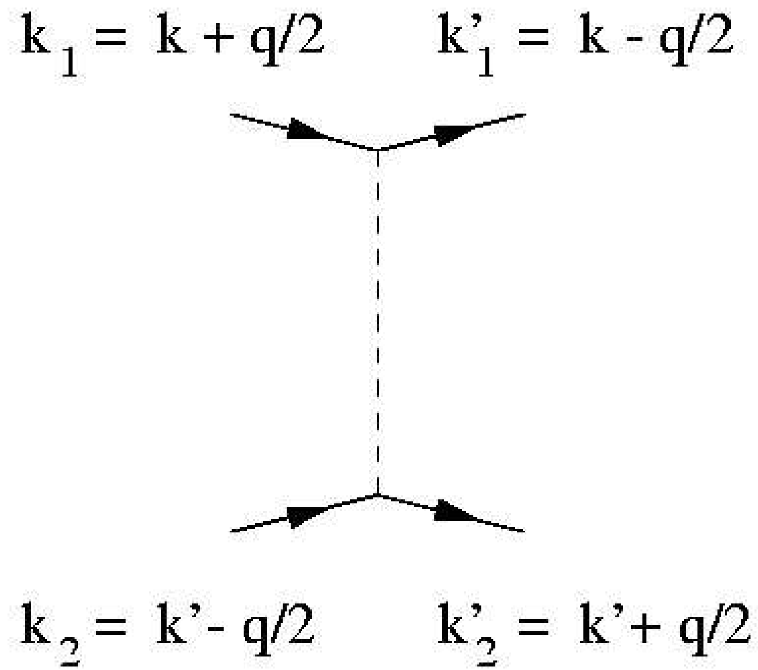
Our objective is to study the properties of interacting system



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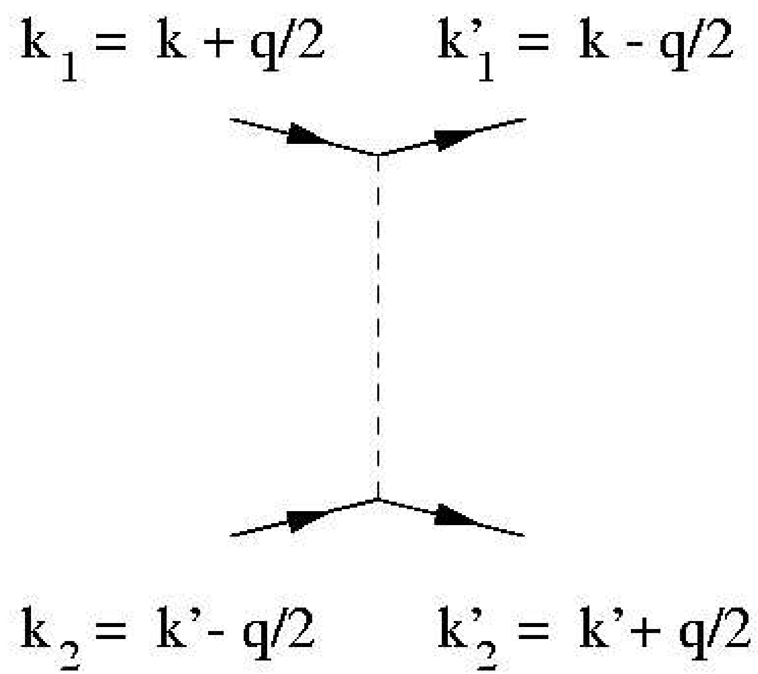

$$\hat{H} = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu) \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \sum_{\sigma,\sigma'} g_{\mathbf{k},\mathbf{k}',\mathbf{q}} \hat{c}^{\dagger}_{\mathbf{k}+\frac{\mathbf{q}}{2},\sigma} \hat{c}^{\dagger}_{\mathbf{k}'-\frac{\mathbf{q}}{2},\sigma'} \hat{c}_{\mathbf{k}'+\frac{\mathbf{q}}{2},\sigma'} \hat{c}_{\mathbf{k}-\frac{\mathbf{q}}{2},\sigma}$$

Functional integral representation

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Thermodynamics of the system (total energy, specific heat, pressure, etc) can be derived from the partition function $\mathcal{Z}$. It can be expressed in terms of the Grassmann variables

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The action $S = S_0 + S_I$ contains the quadratic term (which corresponds to the kinetic energy)

$$S_0 = \int_k \psi^*_{\mathbf{k}} \; (i\omega_n - \epsilon_{\mathbf{k}} + \mu) \; \psi_{\mathbf{k}} \qquad\qquad k \equiv (i\omega_n, \mathbf{k})$$

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and a quartic term (describing the two-body interactions)

$$S_I = -\frac{1}{2} \int_{k,k',q} g_{\mathbf{k},\mathbf{k}',\mathbf{q}} \psi^*_{\mathbf{k}+\frac{\mathbf{q}}{2}} \psi^*_{\mathbf{k}'-\frac{\mathbf{q}}{2}} \psi_{\mathbf{k}'+\frac{\mathbf{q}}{2}} \psi_{\mathbf{k}-\frac{\mathbf{q}}{2}}$$

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where $\chi_{\mathbf{k}}$ and $\chi^*_{\mathbf{k}}$ are the source Grassmann fields.

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For instance, the single particle excitations can be determined from the two-point Green's function

$$\frac{\delta}{\delta \chi^*_{\mathbf{k}}} \frac{\delta}{\delta \chi_{\mathbf{k}}} \mathcal{G}[\chi,\chi^*]|_{\chi=0,\chi^*=0}$$

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A more specific discussion can be found e.g. in

V.N. Popov,
Functional integrals and collective excitations, Cambridge Univ. Press (1987);
J.W. Negele and H. Orland,
Quantum many-particle systems, Perseus Books (1998).

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It is then useful to introduce the *renormalized* action $S^{\Lambda}[\psi, \psi^*]$

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so that the generating functional becomes

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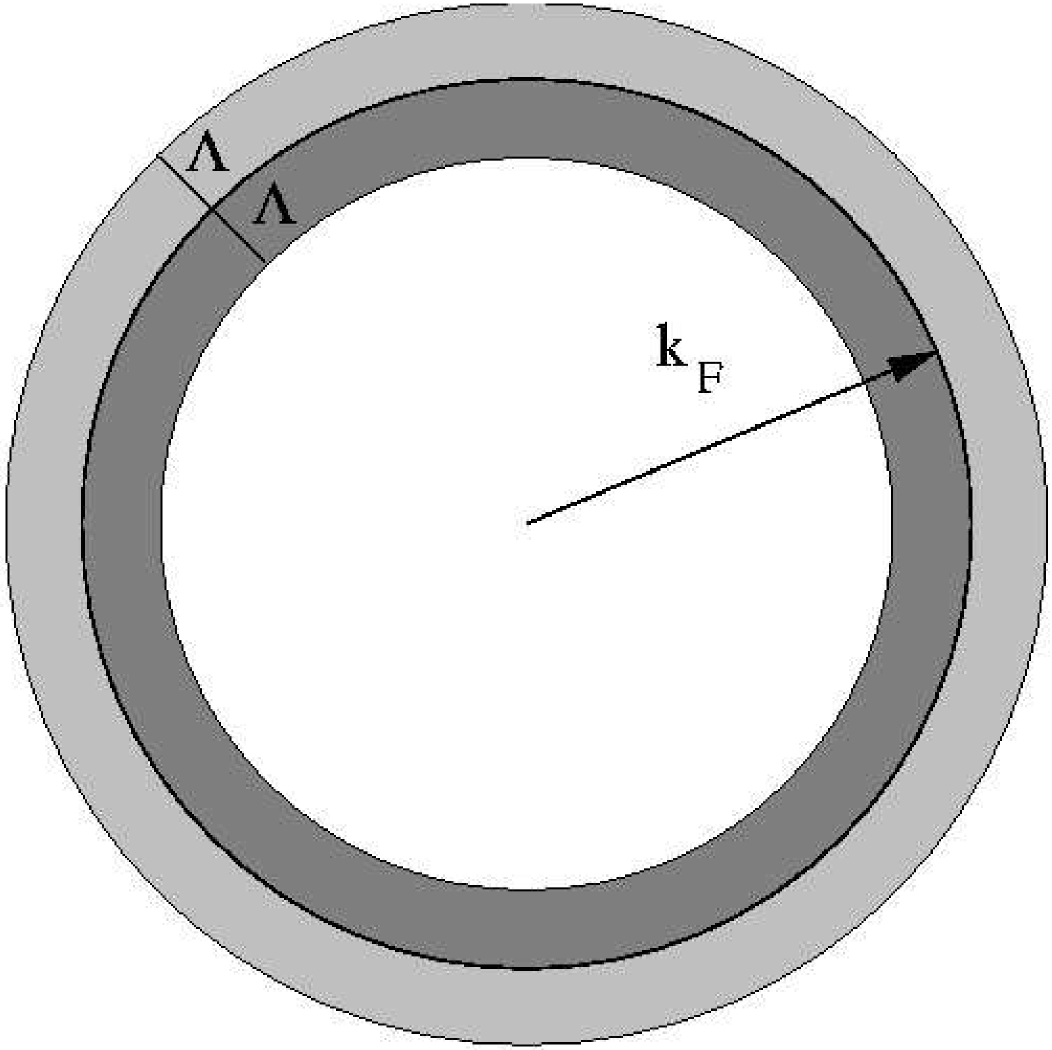
Mode elimination in the momentum space:



Fast modes (i.e. fermion fields outside the shell of width 2Λ) are integrated out and the leftover contains only *slow modes* which are relevant for the physically observed properties.

Nobel Prize in Physics 1982

Kenneth Wilson

for his theory of critical phenomena in connection with phase transitions

Remarks on the RG scheme

Upon a succesive decrease of the energy cut-off Λ toward the Fermi energy the high energy excitations are integrated out. This leads simultaneously to:

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⇒ **position of the Fermi surface might drift**

⇒ **topology of the Fermi surface might deform**

⇒ **and sometimes even the Fermi surface itself might completely break down !**

In case of the symmetry broken phases the scaling procedure is additionally complicated due to the lower boundary (gap Δ).

The conventional RG techniques are blind with respect to the symmetry-broken states which are separated by energy barrier from the symmetric state.

R. Gersch, J. Reiss and C. Honerkamp, Progr. Theor. Phys. (2006).

Possible ways to proceed

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1. A small symmetry-breaking component $\boldsymbol{\Delta(\Lambda_0)}$ is imposed at a certain initial condition Λ_0. Its physical meaning establishes from the flow to the asymptotic fixed point

$$\boldsymbol{\Delta = \lim_{\Lambda \to 0} \Delta(\Lambda)}$$

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2. One introduceds the collective boson field $\boldsymbol{\Phi}$ via Hubbard-Stratonovich transformation and Fermi fields are not completely integrated out but instead of this the effective Fermi-Bose theory is developed using the functional RG equations.

$$\boldsymbol{S[\psi, \psi^*] = S_0[\psi, \psi^*] + S_0[\Phi] + S_I[\psi, \psi^*, \Phi]}$$

F. Schütz, L. Bartosch, P. Kopietz, Phys. Rev. B **72**, *035107 (2005).*

III. New RG formulation

The overall scheme

Instead of integrating out the fast modes (high energy sector) one constructs the canonical transformation $\hat{H}(l) = \hat{U}(l)\hat{H}\hat{U}^{\dagger}(l)$ such that:

- Hamiltonian is diagonalized in a series of infinitesimal steps

$$\hat{H} \longrightarrow \ldots \longrightarrow \hat{H}(l) \longrightarrow \ldots \longrightarrow \hat{H}(\infty)$$

  with l being a continuous parameter

- evolution of the Hamiltonian is governed by **the flow equation**

$$\partial_l \hat{H}(l) = \left[\hat{\eta}(l), \hat{H}(l)\right]$$

  where formally $\hat{\eta}(l) = -\hat{U}(l)\,\partial_l \hat{U}^{\dagger}(l)$.

F. Wegner, Annalen der Physik **3**, *77 (1994).*

Comparison to the RG

Similarities:

- diagonalization of the high energy states occurs mainly during the first part of the transformation
- the low energy states are diagonalized at the very end of transformation

Roughly speaking, one can draw the following relation to the Wilson's numerical RG method:

$$\frac{1}{\sqrt{l}} \longleftrightarrow \Gamma$$

Differences:

Throughout the continuous canonical transformation one keeps track of the slow and high energy modes, therefore their mutual feedback effects can be analyzed.

Practical choice

For Hamiltonians with the structure

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

one can choose

$$\hat{\eta}(l) = \left[\hat{H}_0(l), \hat{H}_1(l)\right]$$

and then

$$\lim_{l\to\infty} \hat{H}_1(l) = 0$$

Other possible ways for constructing the generating operator $\hat{\eta}$ have been discussed by various authors. For a detailed information see for instance:

S. Kehrein, Springer Tracts in Modern Physics **217**, (2006);
F. Wegner, J. Phys. A: Math. Gen. **39**, 8221 (2006).

Correlation functions

To determine the correlation functions

$$\langle \hat{A}(t)\hat{B}(t')\rangle$$

one has to compute the statistical average

$$\langle \ldots \rangle = \mathrm{Tr}\left\{e^{-\beta\hat{H}}\ldots\right\} / \mathrm{Tr}\left\{e^{-\beta\hat{H}}\right\}.$$

This can be done using the invariance

$$\begin{aligned}
\mathrm{Tr}\left\{e^{-\beta\hat{H}}\hat{O}\right\} &= \mathrm{Tr}\left\{e^{\hat{S}(l)}e^{-\beta\hat{H}}\hat{O}e^{-\hat{S}(l)}\right\} \\
&= \mathrm{Tr}\left\{e^{\hat{S}(l)}e^{-\beta\hat{H}}e^{-\hat{S}(l)}e^{\hat{S}(l)}\hat{O}e^{-\hat{S}(l)}\right\} \\
&= \mathrm{Tr}\left\{e^{-\beta\hat{H}(l)}\hat{O}(l)\right\}
\end{aligned}$$

where $\hat{U}(l) \equiv e^{\hat{S}(l)}$ and

$$\hat{H}(l) = e^{\hat{S}(l)}\hat{H}e^{-\hat{S}(l)} \qquad \hat{O}(l) = e^{\hat{S}(l)}\hat{O}e^{-\hat{S}(l)}$$

Some remarks :

- ⋆ The easiest way for calculation is to take the limit $l \longrightarrow \infty$ when $\hat{H}(\infty)$ becomes (block-)diagonal.

- ⋆ However, the observables must be transformed too

$$\hat{O} \longrightarrow \ldots \longrightarrow \hat{O}(l) \longrightarrow \ldots \longrightarrow \hat{O}(\infty)$$

- ⋆ Evolution of the observable is given through *the flow equation*:

$$\partial_l \hat{O}(l) = \left[\hat{\eta}(l), \hat{O}(l)\right]$$

IV. Applications

1. The bilinear Hamiltonian

$$\hat{H} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + \Delta^{*}_{\mathbf{k}} \hat{c}_{-\mathbf{k}\downarrow} \hat{c}_{\mathbf{k}\uparrow} \right)$$

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$$\begin{aligned} \hat{c}_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} \; \hat{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \; \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \\ \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} &= -v_{\mathbf{k}} \; \hat{c}_{\mathbf{k}\uparrow} + u_{\mathbf{k}} \; \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \end{aligned}$$

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N.N. Bogoliubov, Sov. Phys. JETP **7***, 41 (1948)*

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$$\partial_l \hat{H} = \left[\hat{\eta}, \hat{H}\right]$$

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we obtain the set of flow equations

$$\begin{aligned} \partial_l \, \xi_{\mathbf{k}}(l) &= 4\xi_{\mathbf{k}}(l) \, |\Delta_{\mathbf{k}}(l)|^2 \\ \partial_l \, \Delta_{\mathbf{k}}(l) &= -4|\xi_{\mathbf{k}}(l)|^2 \, \Delta^*_{\mathbf{k}}(l) \end{aligned}$$

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which yield the following identity

$$|\Delta_{\mathbf{k}}(l)| = |\Delta_{\mathbf{k}}| e^{-4\int_0^l dl' [\xi_{\mathbf{k}}(l')]^2}.$$

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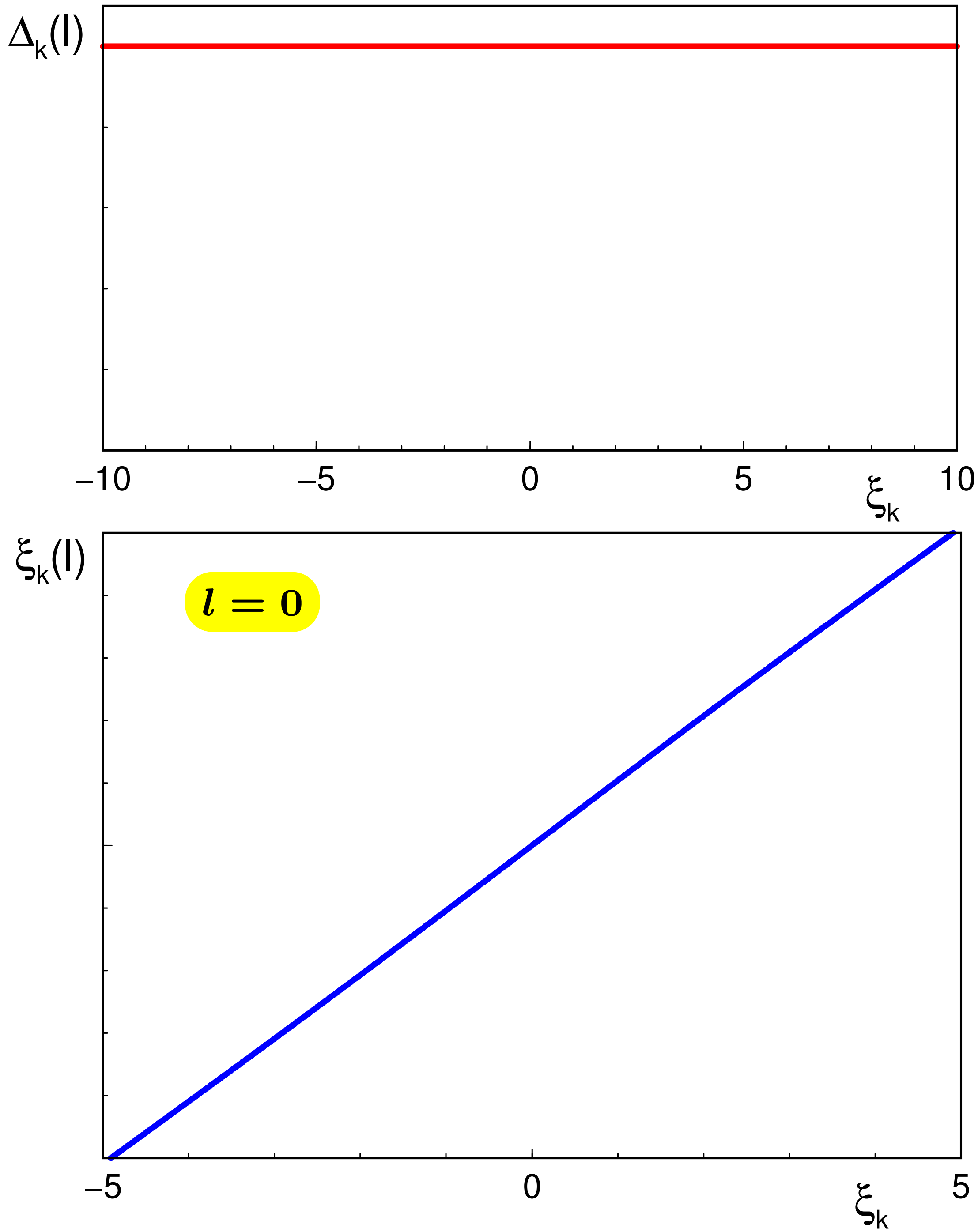
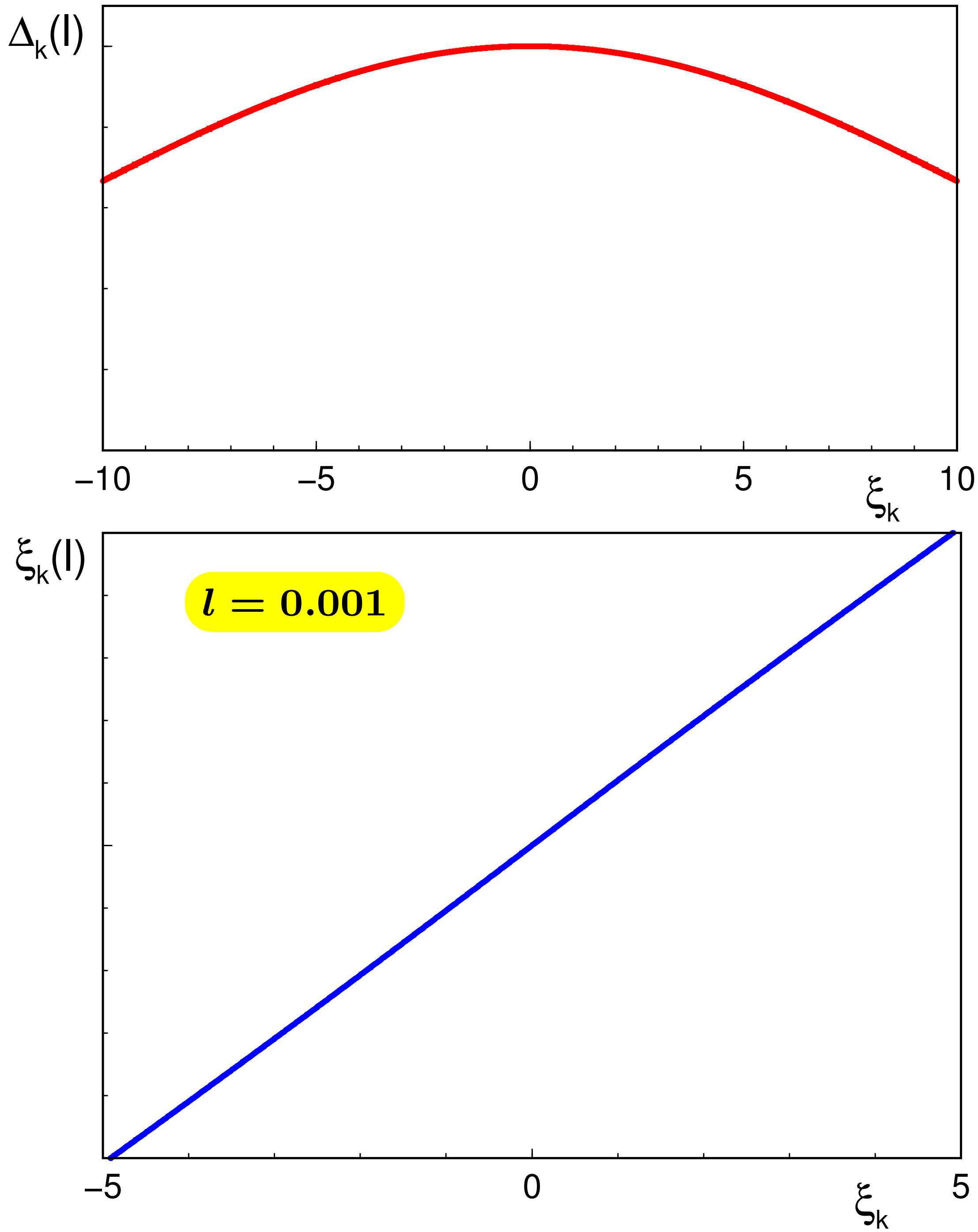
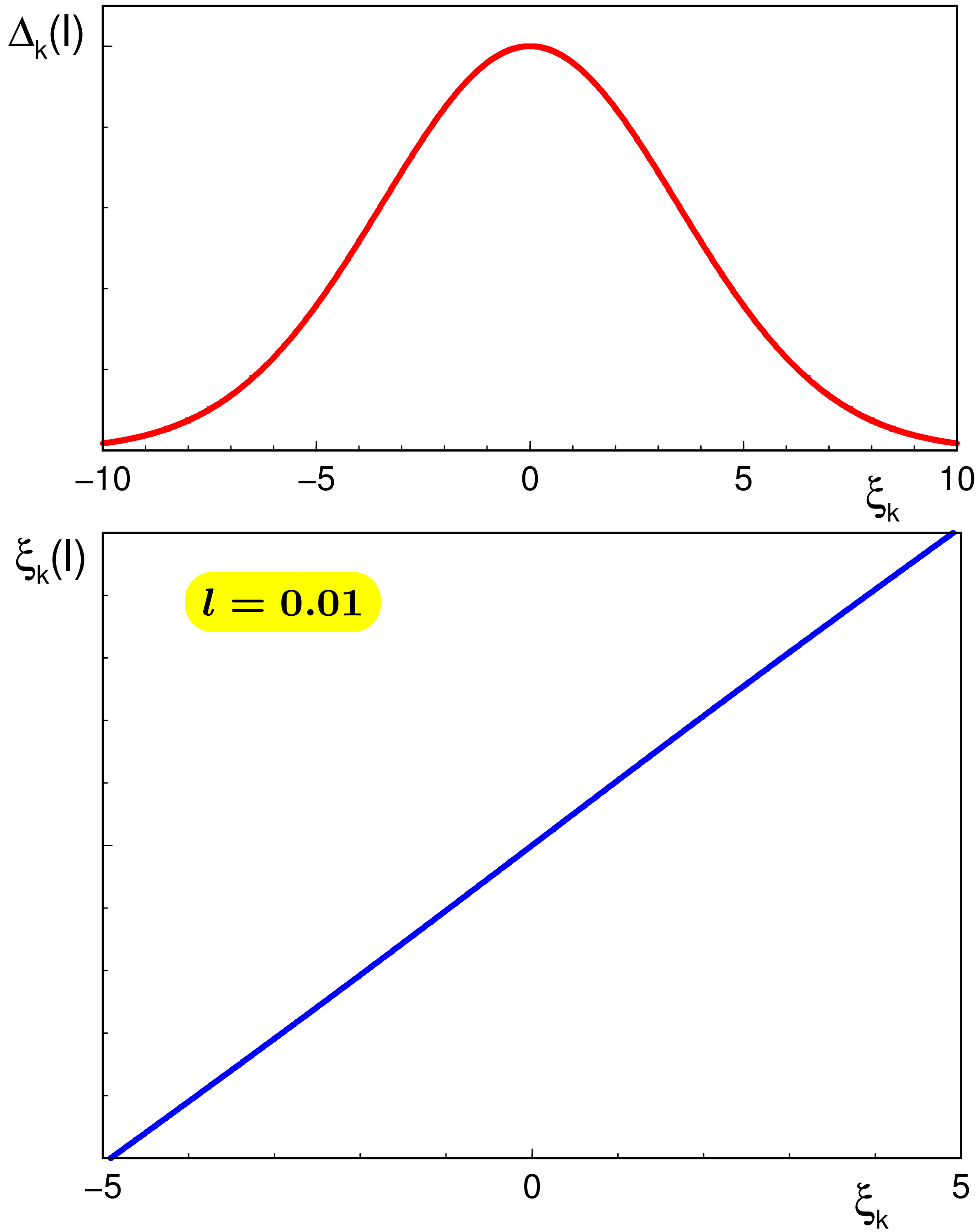
T. Domański, cond-mat/0602236 (to be published).



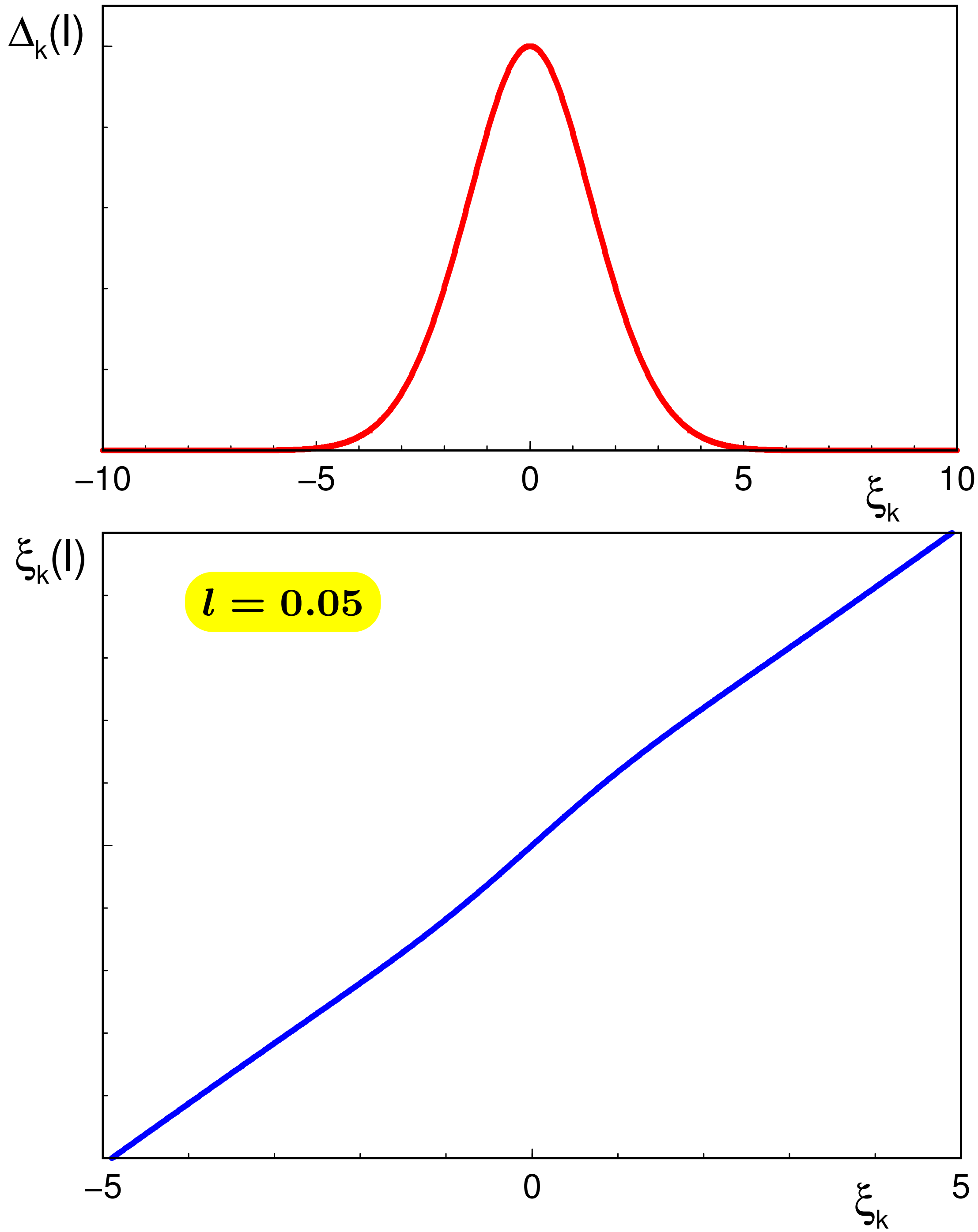
Δk(l)
–10
–5
0
5
10
ξk
ξk(l)
l = 0
–5
0
5
ξk



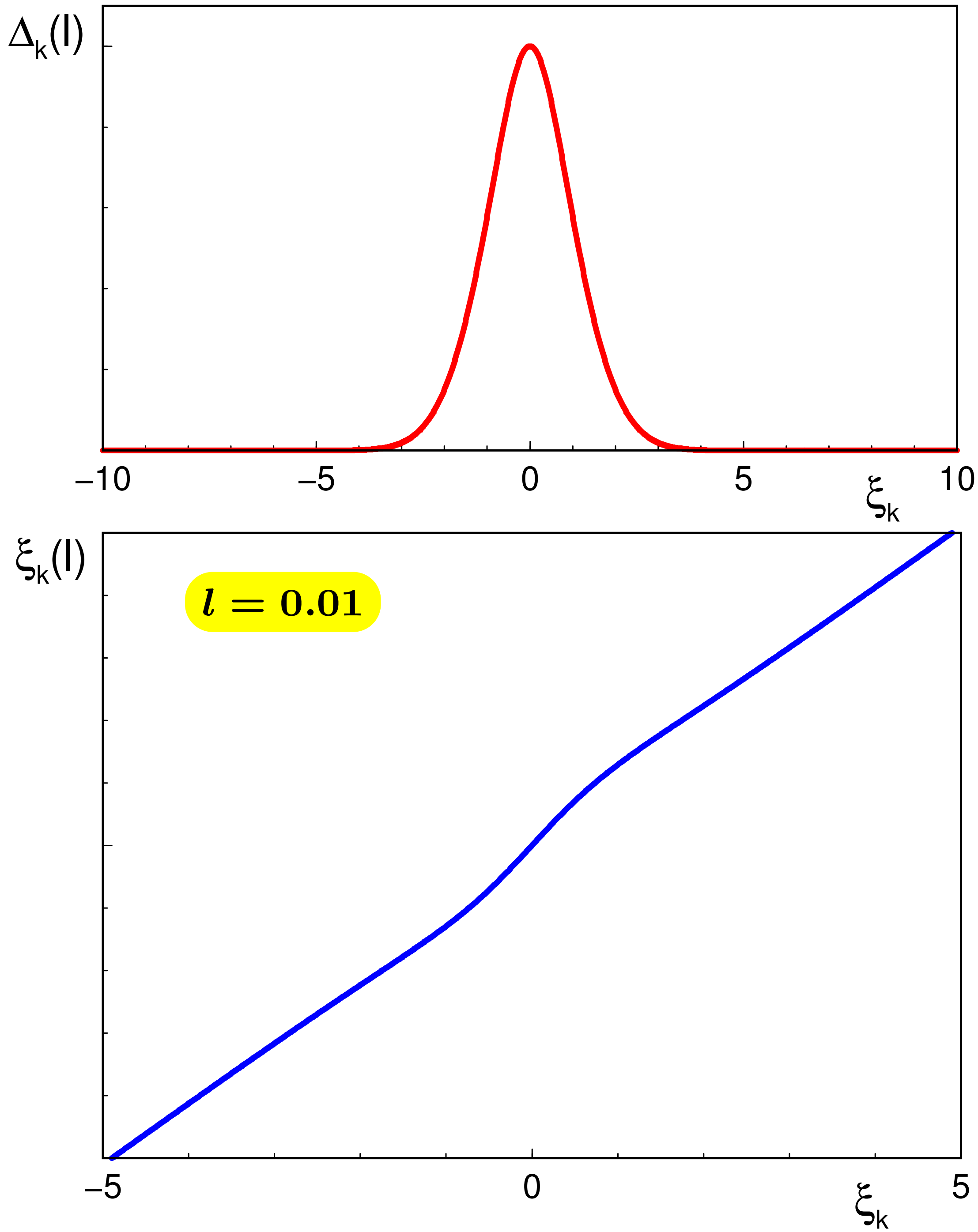
Δk(l)
−10
−5
0
5
10
ξk
ξk(l)
l = 0.001
−5
0
5
ξk



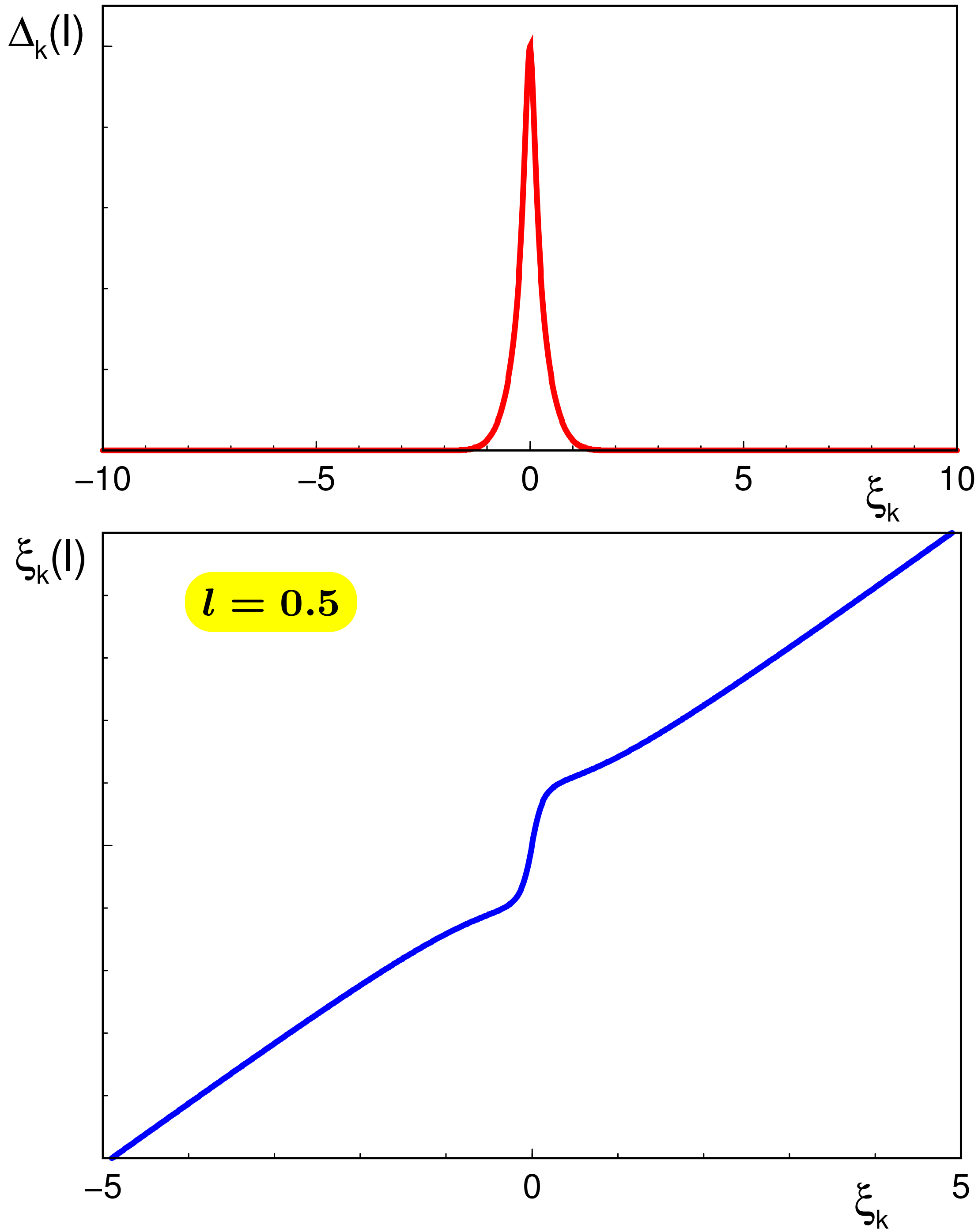
Δk(l)
−10
−5
0
5
10
ξk
ξk(l)
l = 0.01
−5
0
5
ξk



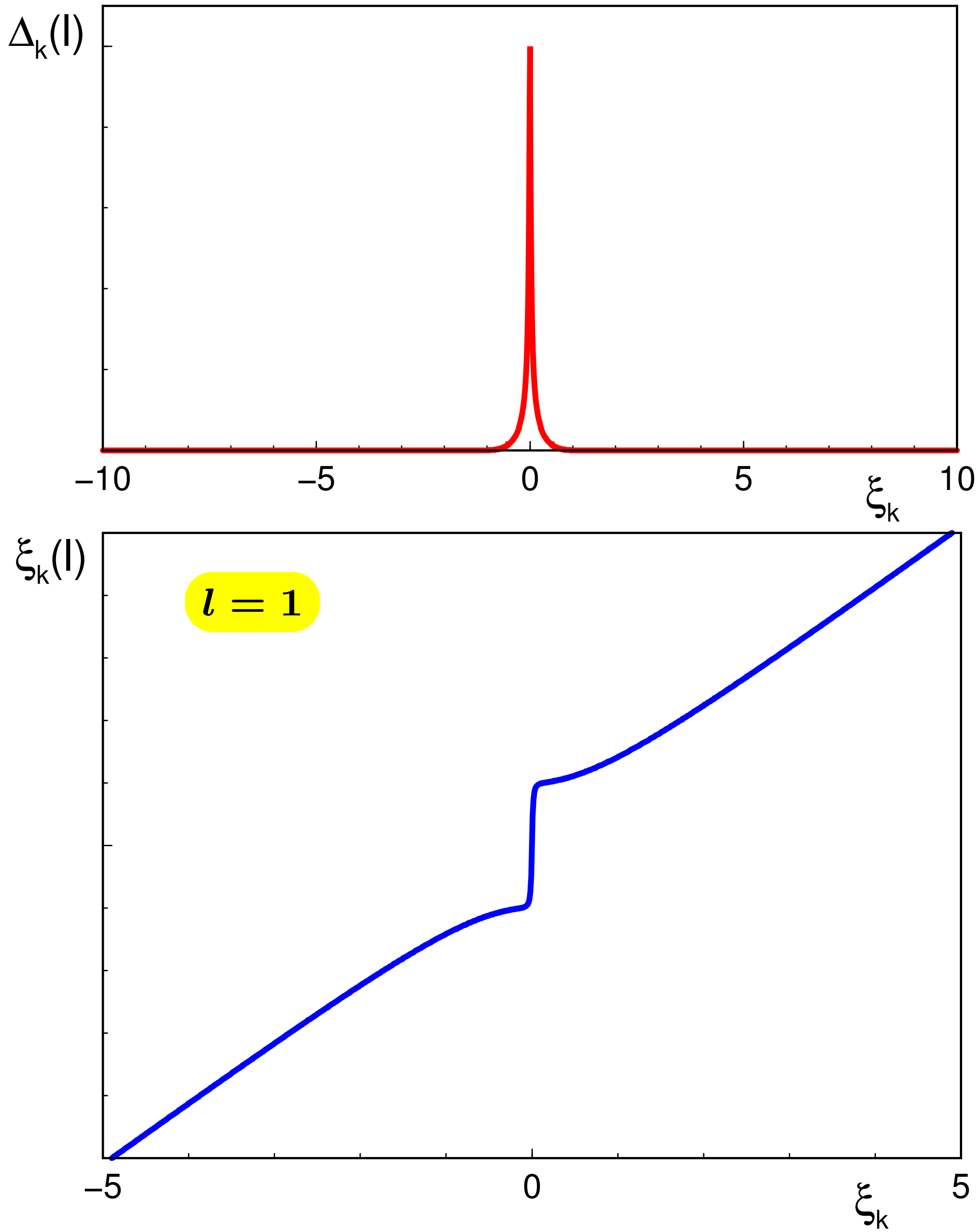
$\Delta_k(l)$
−10
−5
0
5
10
ξ_k
$\xi_k(l)$
$l = 0.05$
−5
0
5
ξ_k



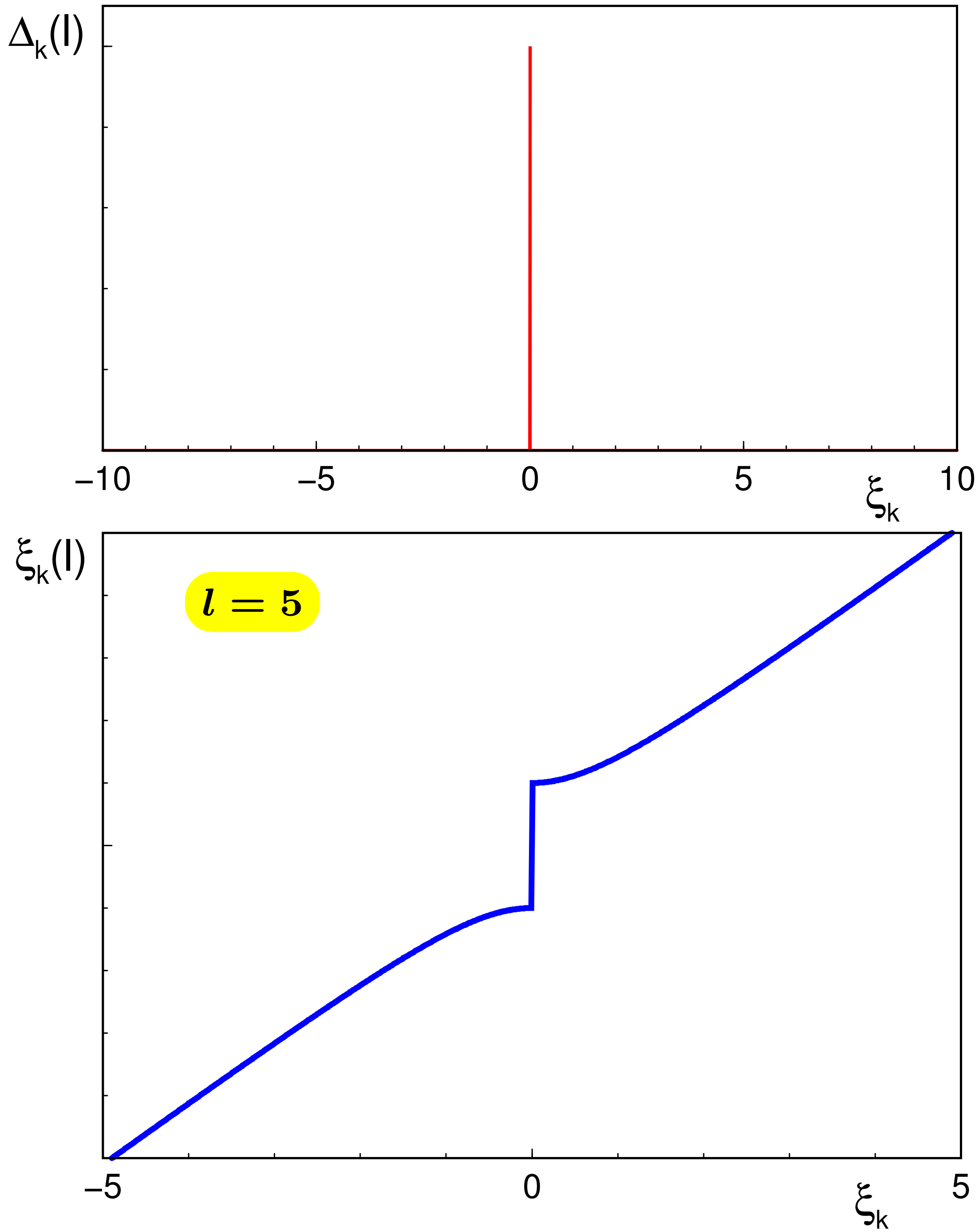
Δk(l)
–10
–5
0
5
10
ξk
ξk(l)
l = 0.01
–5
0
5
ξk



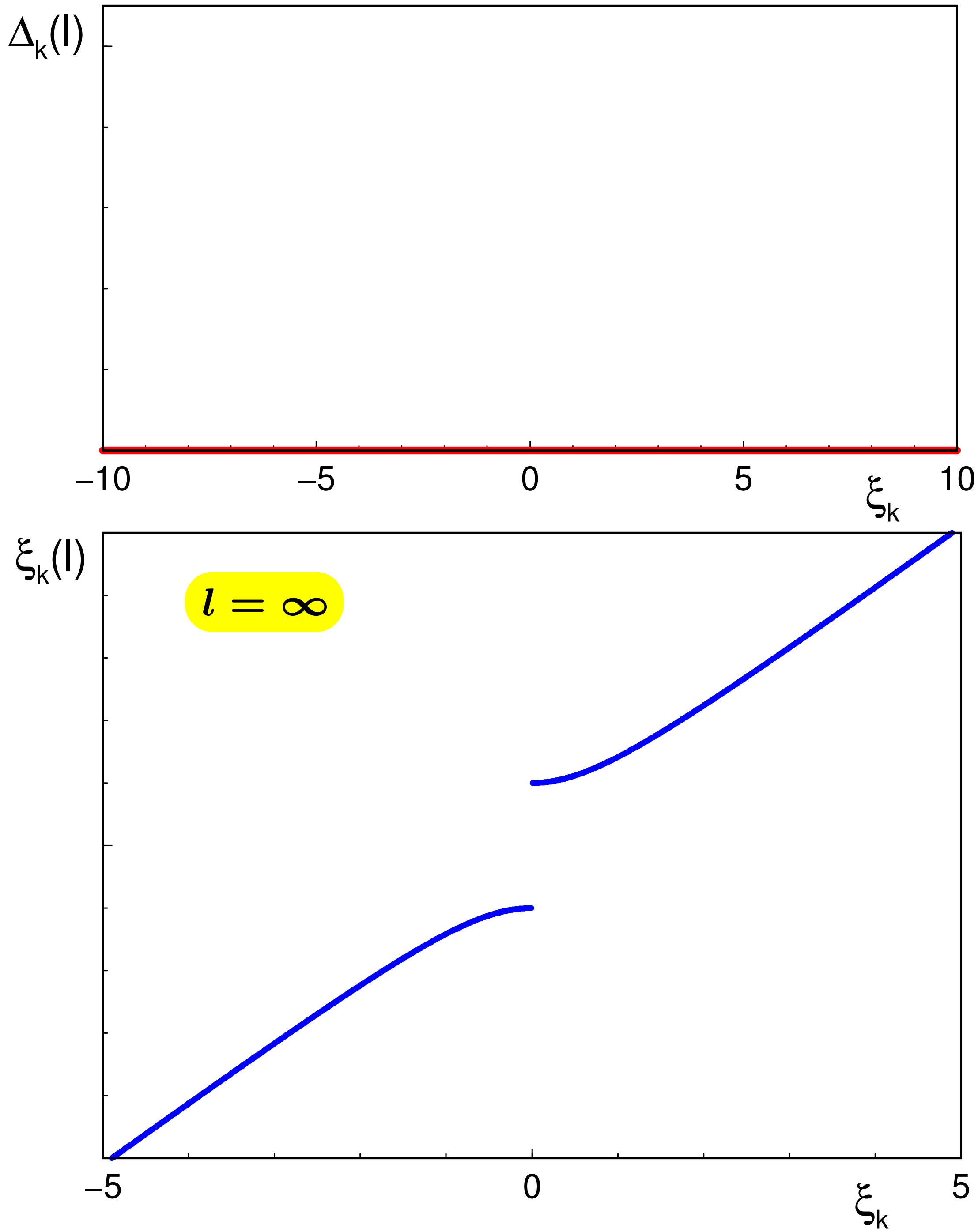
Δk(l)
−10
−5
0
5
10
ξk
ξk(l)
l = 0.5
−5
0
5
ξk



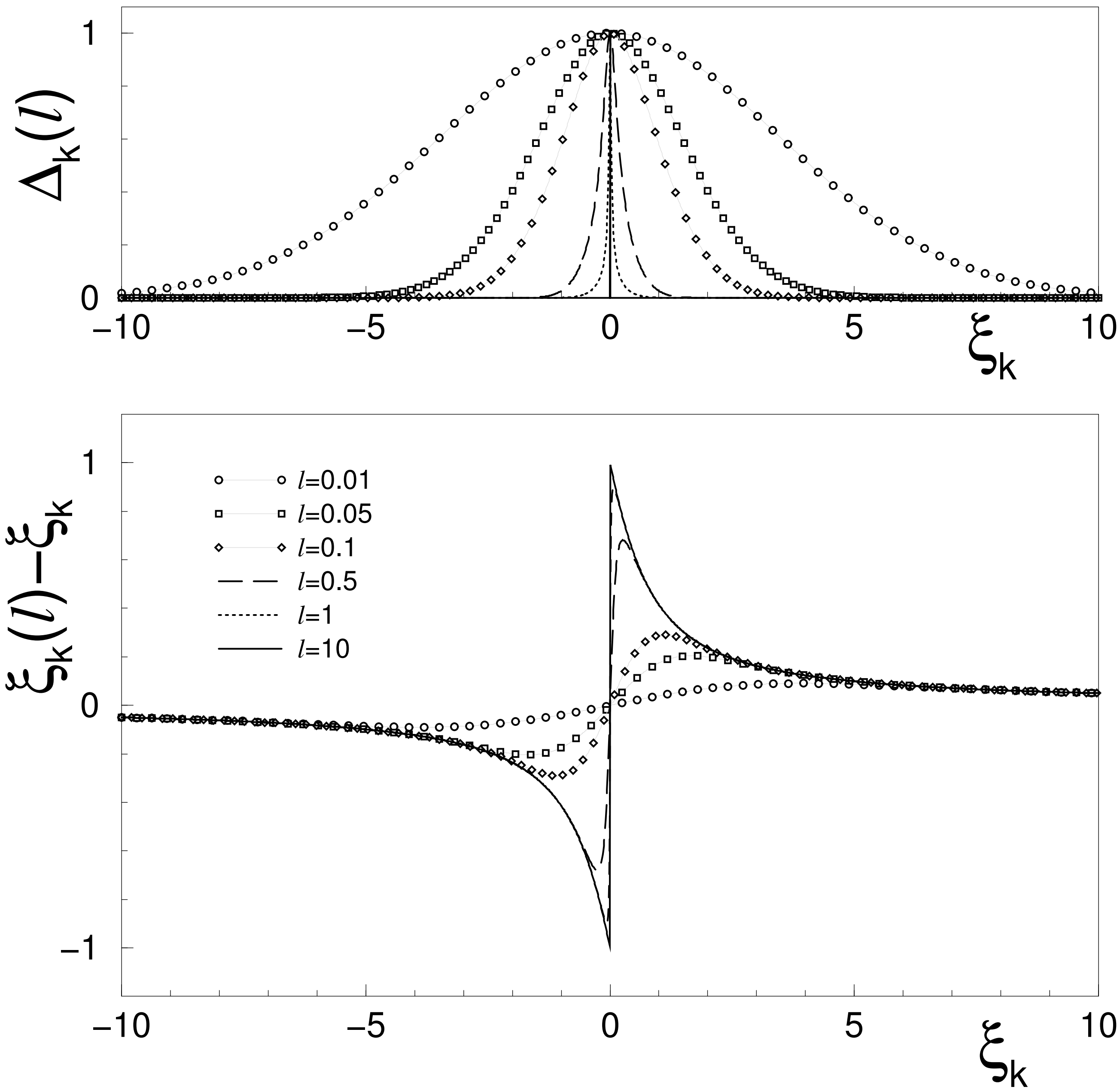
Δk(l)
−10
−5
0
5
10
ξk
ξk(l)
l = 1
−5
0
5
ξk



Δk(l)
−10
−5
0
5
10
ξk
ξk(l)
l = 5
−5
0
5
ξk



$\Delta_k(l)$
–10
–5
0
5
10
ξ_k
$\xi_k(l)$
$l = \infty$
–5
0
5
ξ_k



Renormalization of $\Delta_{\mathbf{k}}(l)$ and $\xi_{\mathbf{k}}(l)$ during the flow.

For determination of the single particle spectrum and the order parameter we must analyze evolution of the fermion operators.

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The $l = 0$ derivatives yield the Ansatz

$$\begin{aligned} \hat{c}_{\mathbf{k}\uparrow}\ (l) &= u_{\mathbf{k}}(l)\ \hat{c}_{\mathbf{k}\uparrow}\ (l) + v_{\mathbf{k}}(l)\ \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}(l) \\ \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}(l) &= -v_{\mathbf{k}}(l)\ \hat{c}_{\mathbf{k}\uparrow}\ (l) + u_{\mathbf{k}}(l)\ \hat{c}^{\dagger}_{-\mathbf{k}\downarrow}(l) \end{aligned}$$

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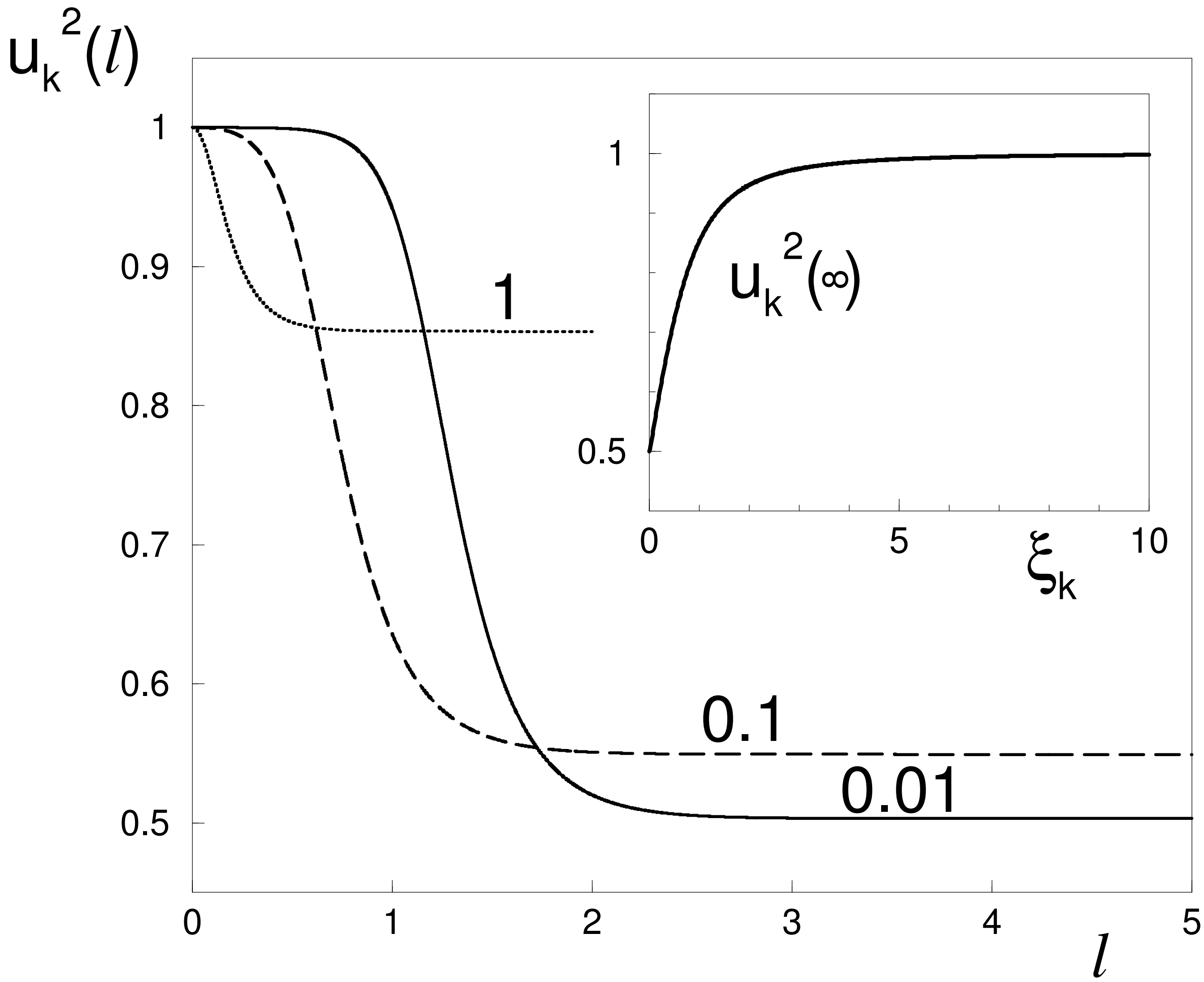
where *flow* of the coefficients is given by

$$\begin{aligned} \partial_l\, u_{\mathbf{k}}(l) &= 2\xi_{\mathbf{k}}(l)\, \Delta_{\mathbf{k}}(l)\, v_{\mathbf{k}}(l) \\ \partial_l\, v_{\mathbf{k}}(l) &= -2\xi_{\mathbf{k}}(l)\, \Delta_{\mathbf{k}}(l)\, u_{\mathbf{k}}(l) \end{aligned}$$



Flow of the coherence factor $u_{\mathbf{k}}^2(l)$ for $\xi_{\mathbf{k}}/\Delta = 0.01, 0.1$ and 1.

2. The boson-fermion model

$$
\begin{aligned}
H &= \sum_{\mathbf{k}\sigma} (\varepsilon_{\mathbf{k}} - \mu)\, c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} (E_{\mathbf{q}} - 2\mu)\, b^{\dagger}_{\mathbf{q}} b_{\mathbf{q}} \\
&+ \frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{q}} v_{\mathbf{k},\mathbf{q}} \left[b^{\dagger}_{\mathbf{q}} c_{\mathbf{k},\downarrow} c_{\mathbf{q}-\mathbf{k},\uparrow} + \text{h.c.} \right]
\end{aligned}
$$

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$$b^{\dagger}_{\mathbf{q}} \equiv \sum_{\mathbf{k}'} c^{\dagger}_{\mathbf{q}-\mathbf{k}',\uparrow}\; c^{\dagger}_{\mathbf{k}',\downarrow}$$

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The mean-field approximation

$$H \simeq \sum_{\mathbf{k}\sigma} (\varepsilon_{\mathbf{k}} - \mu)\, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} (E_{\mathbf{q}} - 2\mu)\, b^\dagger_{\mathbf{q}} b_{\mathbf{q}}$$

$$+ \frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{q}} v_{\mathbf{k},\mathbf{q}} \left[\langle b^\dagger_{\mathbf{q}} \rangle c_{\mathbf{k},\downarrow} c_{\mathbf{q}-\mathbf{k},\uparrow} + b^\dagger_{\mathbf{q}} \langle c_{\mathbf{k},\downarrow} c_{\mathbf{q}-\mathbf{k},\uparrow} \rangle + \text{h.c.} \right]$$

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Hamiltonian at $\mathbf{0 < l < \infty}$

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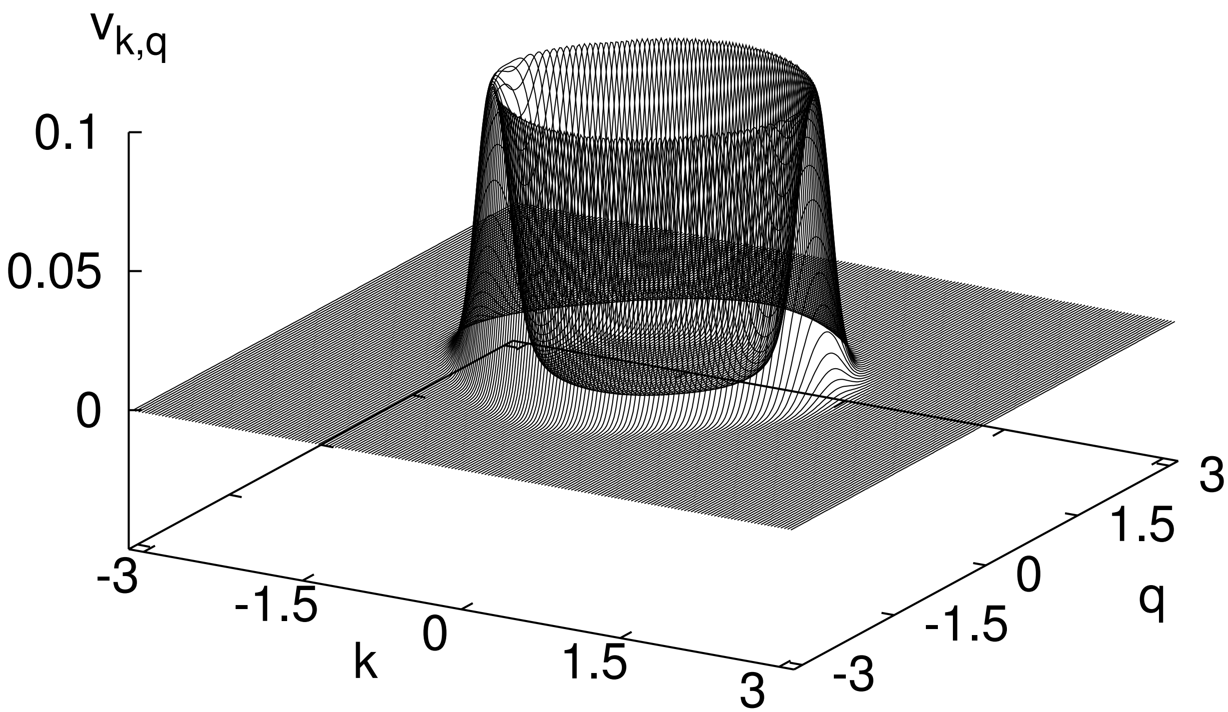
Hamiltonian at $l = \infty$

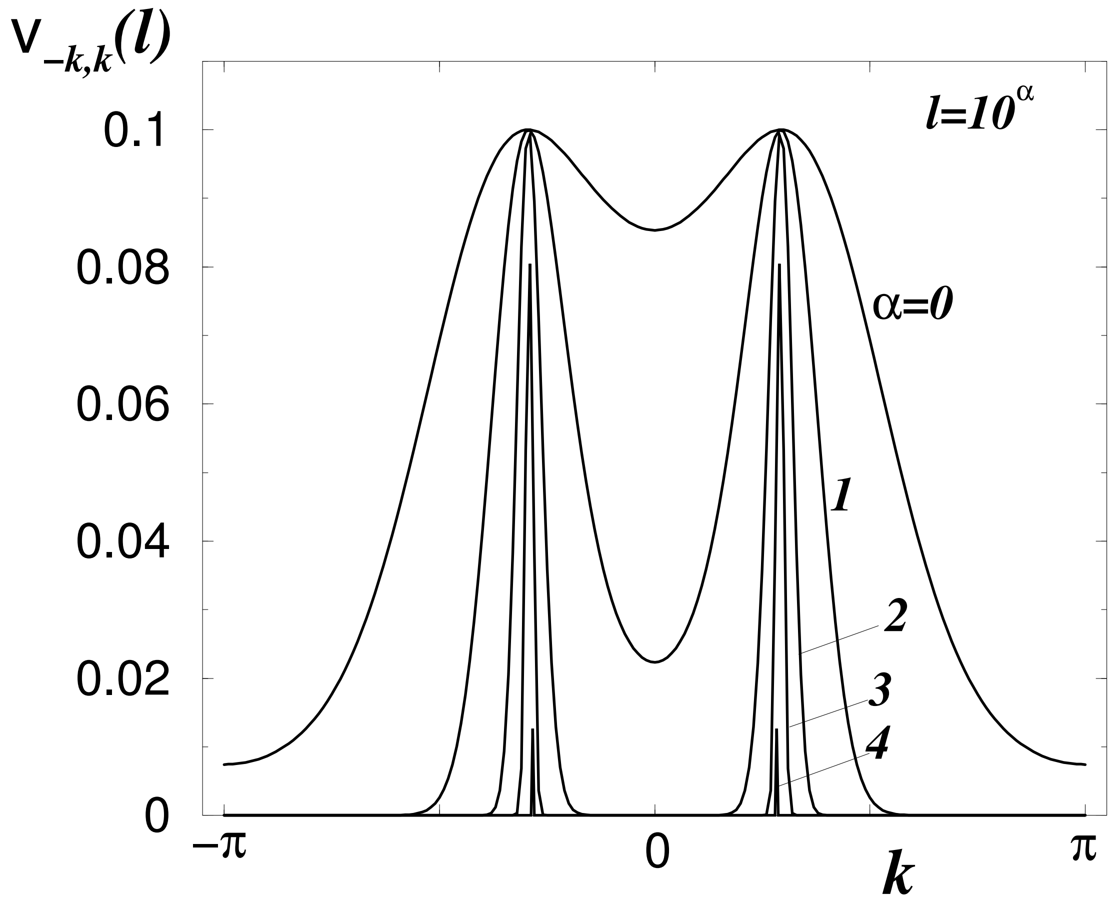
$$\hat{H}_F(\infty) + \hat{H}_B(\infty)$$

The boson-fermion coupling $v_{\mathbf{k},\mathbf{q}}(l)$ during the flow.



T. Domański, J. Ranninger, Phys. Rev. **B 63**, *134505 (2001).*

Flow of the boson-fermion coupling element $v_{-\mathbf{k},\mathbf{k}}(l)$.



T. Domański, J. Ranninger, Phys. Rev. **B 63**, 134505 (2001).

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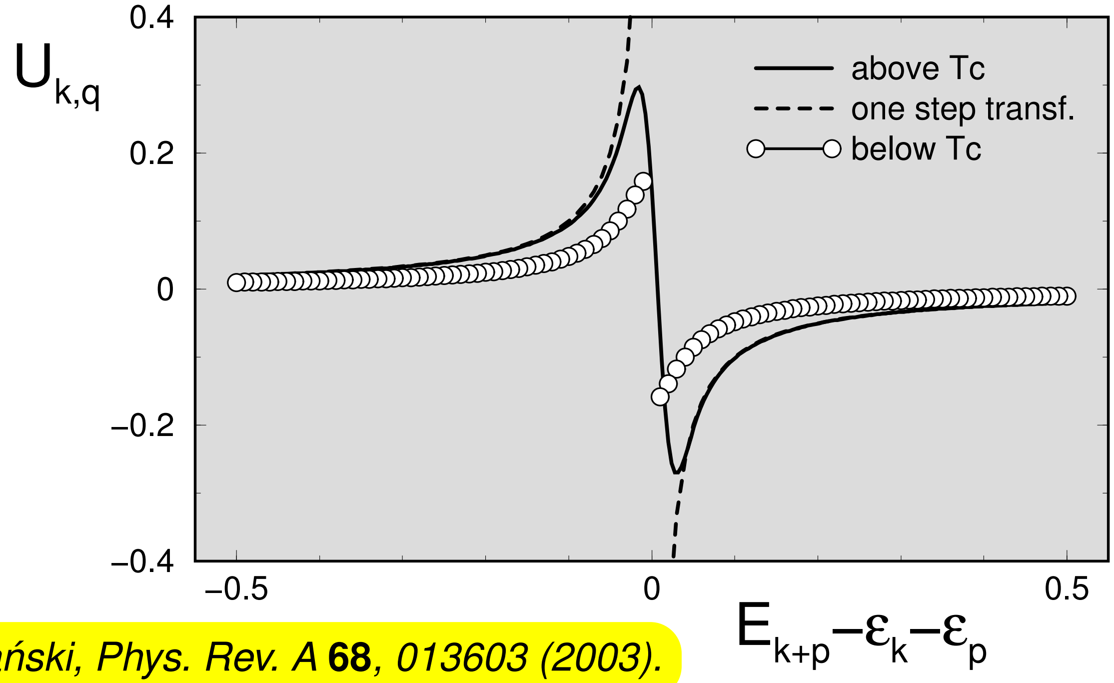
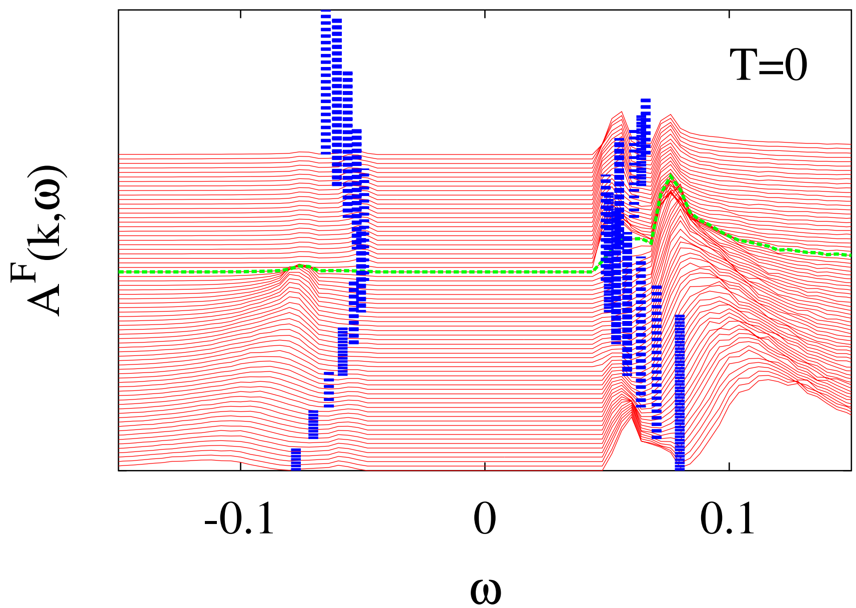


T. Domański, Phys. Rev. A ***68****, 013603 (2003).*

c) The single particle spectrum

Bogoliubov-like spectrum

$T < T_c$

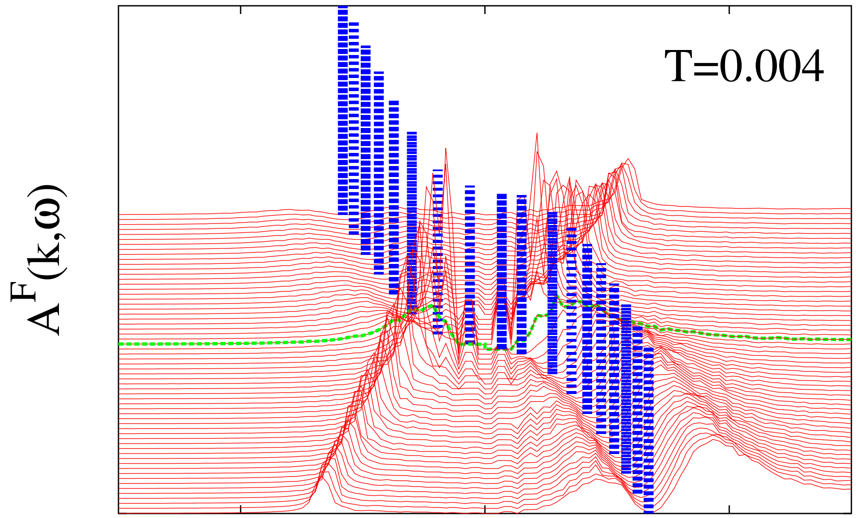


Below the critical temperature T_c *there exist two branches of the excitations at energies* $\omega = \pm\sqrt{(\varepsilon_{\mathbf{k}} - \mu)^2 + \Delta_{sc}^2}$ *(like in the BCS theory).*

T. Domański and J. Ranninger, Phys. Rev. Lett. **91**, *255301 (2003).*

Bogoliubov-like spectrum

$T < T^{*}$



*Pasing through T_c the Bogoliubov-type spectrum survives but one branch (*the shaddow*) gets damped. Physically it means that fermion pairs no longer have an infinite life-time.*

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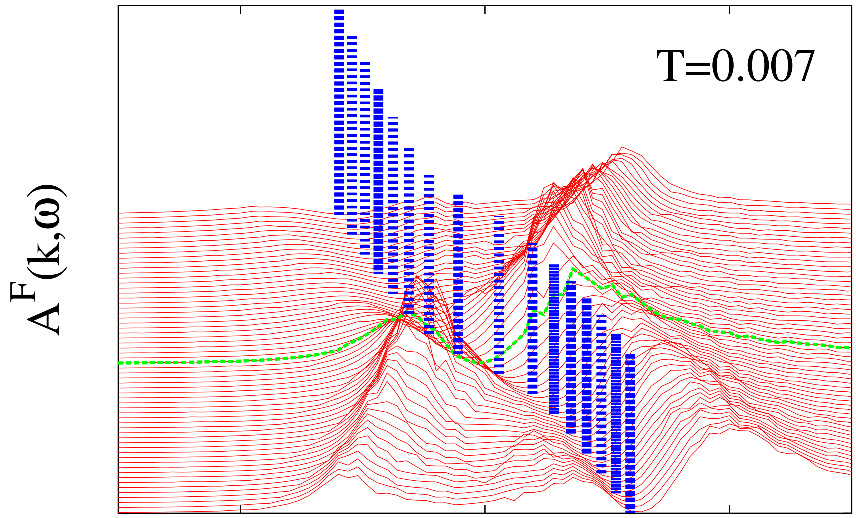

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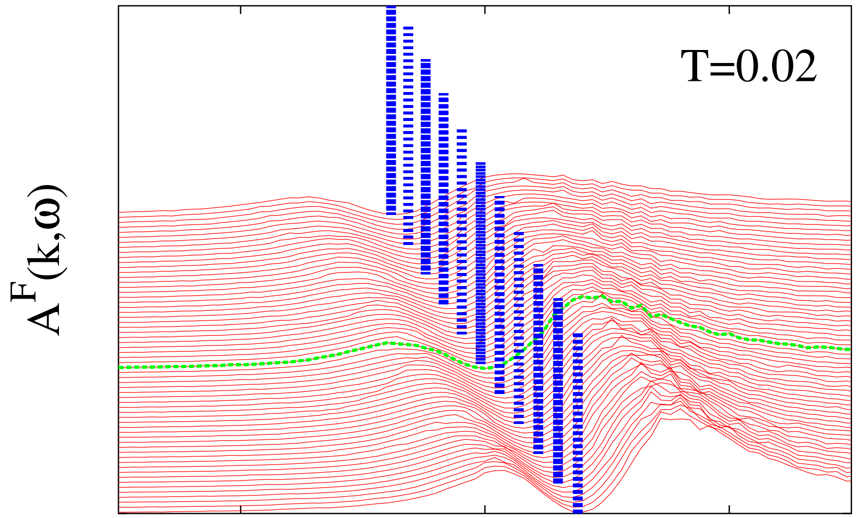

For temperatures far above T_c the Bogoliubov modes are completely gone. There remains only one well established single quasiparticle peak without a gaped dispersion.

T. Domański and J. Ranninger, Phys. Rev. Lett. **91***, 255301 (2003).*

d) Collective phenomena

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Investigating the correlation function of the fermion pairs

$$\left\langle \sum_{\mathbf{k}} c_{\mathbf{k}\downarrow}(\tau) c_{\mathbf{q}-\mathbf{k}\uparrow}(\tau) \sum_{\mathbf{k}'} c^{\dagger}_{\mathbf{q}-\mathbf{k}'\uparrow}(\tau') c^{\dagger}_{\mathbf{k}'\downarrow}(\tau') \right\rangle$$

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we found that the corresponding spectral function

$$\mathcal{N}_{\mathbf{q}}\, \delta\left(\omega - \tilde{E}_{\mathbf{q}}\right) + \mathcal{A}^{inc}_{\mathbf{k}}(\omega)$$

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- and the incoherent background $\mathcal{A}^{inc}_{\mathbf{k}}(\omega)$.

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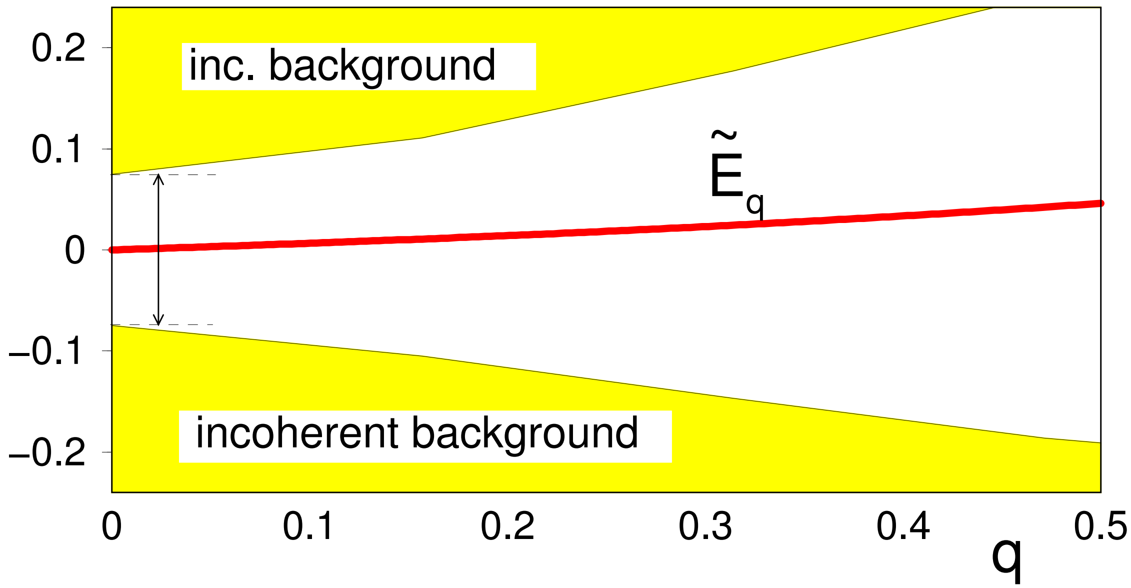
T. Domański and J. Ranninger, Phys. Rev. B **70**, *184513 (2004).*

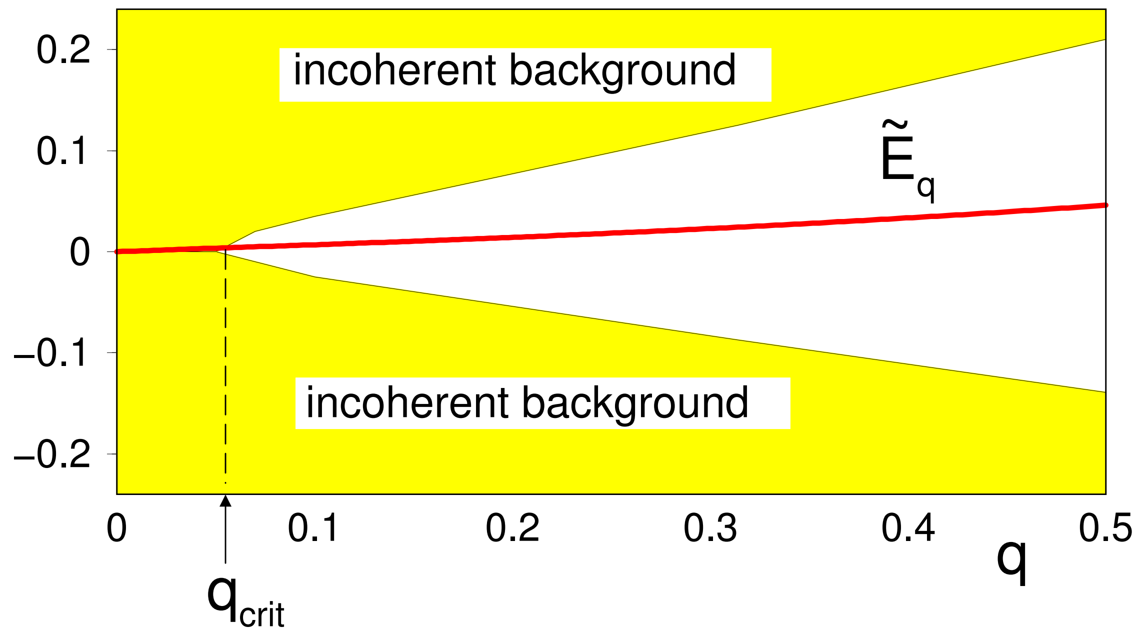
The pair spectrum for $T < T_c$



The quasiparticle peak is well separated from the incoherent background and, in the limit $\mathbf{q} \to \mathbf{0}$, has a characteristic dispersion $\tilde{E}_{\mathbf{q}} = c\,|\mathbf{q}|$. This Goldstone mode is a hallmark of the symmetry broken state.

Such a unique situation could be observed in the case of ultracold fermion atoms, otherwise the Coulomb repulsions lift this mode to the high plasmon frequency.

The pair spectrum for $T^* > T > T_c$



Above the transition temperature (for $T > T_c$):

- *the qusiparticle peak overlaps at small momenta with the incoherent background,*
- *for $\mathbf{q} \to \mathbf{0}$ the Goldstone mode disappears,*
- *remnant of the Goldstone mode is seen above $\mathbf{q}_{crit}$.*

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- **Technical remarks**

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★ **Physical remarks**

Formation of the fermion pairs is usually accompanied by appearance of superfluidity/superconductivity.

Strong quantum fluctuations can suppress the long-range coherence (ordering) while fermion pairs are preserved.